Simplified Settings for Discrete Logarithms in Small Characteristic Finite Fields

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Joint work with Cécile Pierrot

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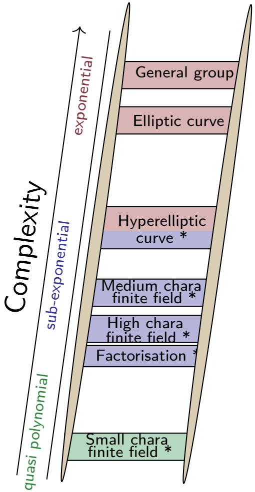

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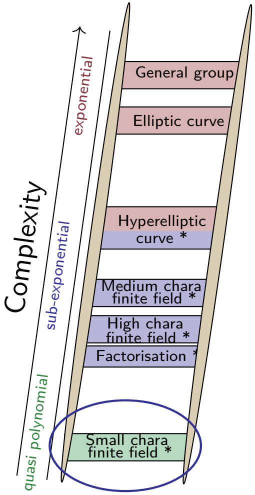

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    - Generic algorithms (Pollard's Rho, Pohlig-Hellman...)
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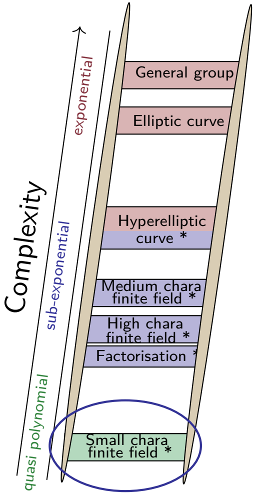

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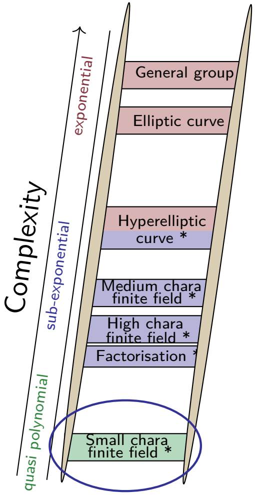

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- Given $|G| = \prod_{i=1}^{k} p_i^{e_i}$
- To compute dlogs in G, it suffices to compute dlogs in:

$$G_i = \langle g^{|G|/p_i} \rangle \quad \text{(Group of order } p_i\text{)}$$

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- Can be improved to memoryless algorithms using cycle finding techniques

Index Calculus Algorithms

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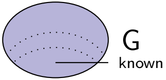

1. Collection of Relations
   → Create a lot of sparse multiplicative relations between some (small) specific elements = the factor base

$$\prod g_i^{e_i} = \prod g_i^{e'_i} \quad \Rightarrow \quad \sum (e_i - e'_i) \log(g_i) = 0$$

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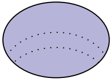
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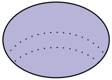

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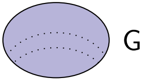

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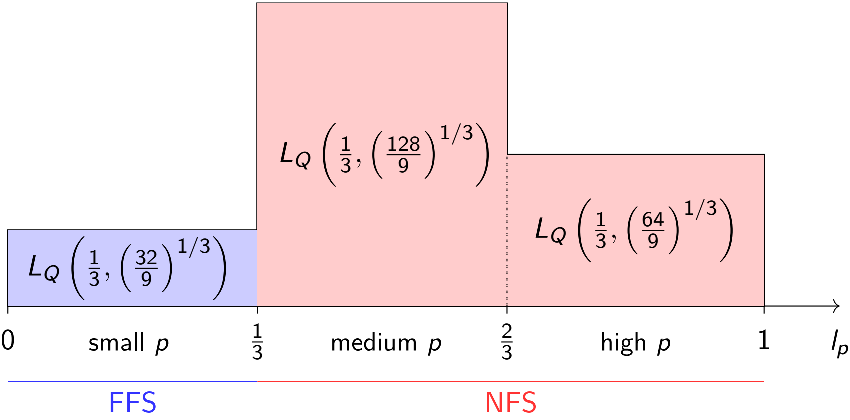
4. Individual Logarithm Phase

   $\rightarrow$ Recover the Discrete Log of an arbitrary element

Complexity of Index calculus algorithms (before 2013)

$$L_Q(\beta, c) = \exp((c + o(1))(\log Q)^{\beta}(\log \log Q)^{1-\beta}).$$



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- In this talk:
  Simplified description of algorithms + additional ideas
  $\Rightarrow$ Improve the complexity of the polynomial phases.

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$$\prod_{\alpha \in \mathbb{F}_q} (X - \alpha) = X^q - X \quad \text{and} \quad \underbrace{X^q = \frac{h_0(X)}{h_1(X)}}_{\text{Frobenius Representation}} \quad \text{or} \quad \underbrace{X = \frac{h_0(X^q)}{h_1(X^q)}}_{\text{Dual Frob. Rep.}}$$

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h_0 :	deg 1 polynomial	or	h_0 :	deg 2 polynomial	useful variant
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Let A and B be 2 small polynomials in $\mathbb{F}_q[X]$ (i.e. of degree $\leqslant D$).

$$B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X)) \underset{\text{thanks to } (**)}{=} A(X)^q B(X) - A(X) B(X)^q$$

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We finally get:

$$\underbrace{h_1(X)^D B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X))}_{\text{Product of small polynomials !!}} = [A, B]_D(X)$$

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- In the sequel, we assume:

$$\begin{aligned} A(X) &= X^D + A_{D-2}(X) \text{ and} \\ B(X) &= X^{D-1} + B_{D-2}(X). \end{aligned}$$

A Small Factor Base

We have: $$\underbrace{h_1(X)^D B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X))}_{\text{polynomials of degree} \leqslant D} = [A, B]_D(X)$$

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- $D \searrow \Rightarrow$ size of the factor base $\searrow \Rightarrow$ complexity of Linear Algebra $\searrow$. The smaller, the better.
- What is simple ? Irreducible poly in $\mathbb{F}_q[X]$ of degree $\leqslant 2$.
- Yet, lowering D rises 2 problems:
  1. Need to generate enough good equations = equations where $[A, B]_2$ splits in terms of degree $\leqslant 2$. Pb: the probability $\mathcal{P}$ to have good equations is too small w.r.t the number of equations required (need $\mathcal{P} > 1/2$).
  2. Need to be able to descend large polynomials to degree 2 ones.

A Small Factor Base: Systematic factors of $[A, B]_D$

- First goal, solving pb 1: i.e. improve the probability $\mathcal{P}$.
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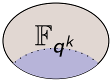

$\Rightarrow$ Linear Algebra permits to recover the DLogs of the factor base in $O((\underbrace{\#\text{ factor base}}_{q^2})^2(\underbrace{\#\text{ of entries}}_{q})) \approx O(q^5)$ operations.

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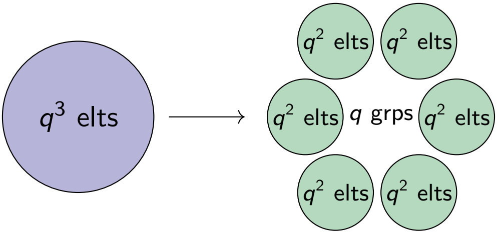

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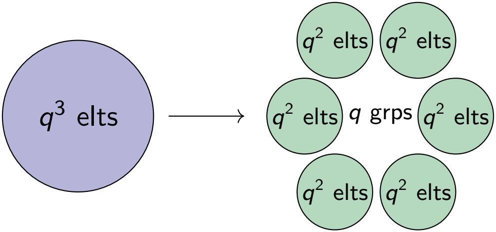

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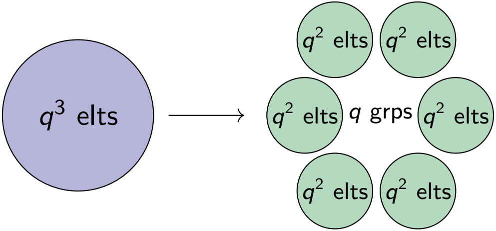

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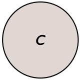
2. Given q^2, generate equations involving only poly in q^2 and degree 1 and 2 polynomials (whose logs are already known).

Extend the Factor Base to Degree 3

- An example: let $\textcircled{c} = \{(X^3 + c) + \alpha X^2 + \beta X | (\alpha, \beta) \in \mathbb{F}_q{}^2\}$.



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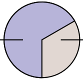
Reducible  Irreducible $\Rightarrow$ new unknowns

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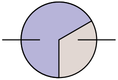
Reducible  Irreducible $\Rightarrow$ new unknowns

As for degree 2: set $A(X) = (X^3 + c) + \alpha X^2$ and $B(X) = (X^3 + c) + \beta X$ and create relations of the form:

$$\underbrace{h_1(X)^3 B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X))}_{\text{all belongs to } \textcircled{c} \text{ !!}} \quad = \quad \underbrace{[A, B]_3(X)}$$

deg 8 with these A and B
$+$ deg 3 systematic factor
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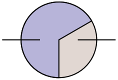
Reducible  Irreducible $\Rightarrow$ new unknowns

As for degree 2: set $A(X) = (X^3 + c) + \alpha X^2$ and $B(X) = (X^3 + c) + \beta X$ and create relations of the form:

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+ deg 3 systematic factor
+ divisible by X

Prob that $[A, B]_3$ factors into deg $\leqslant 2 \Rightarrow 41\%$. Enough !

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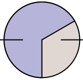
Reducible  Irreducible $\Rightarrow$ new unknowns

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- Complexity to recover the Dlogs of all degree 3 polynomials:

$$O((\underbrace{\# \textcircled{c}}_{q})(\underbrace{\# \text{ factor base}}_{q^2})^2(\underbrace{\# \text{ of entries}}_{q})) \approx O(q^6) \text{ ops.}$$

Extend the Factor Base to Degree 4

Third goal: extend the factor base to degree 4
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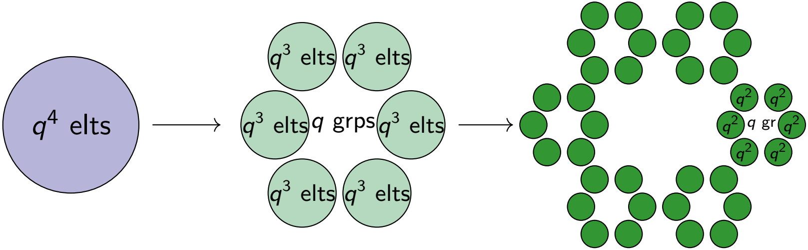

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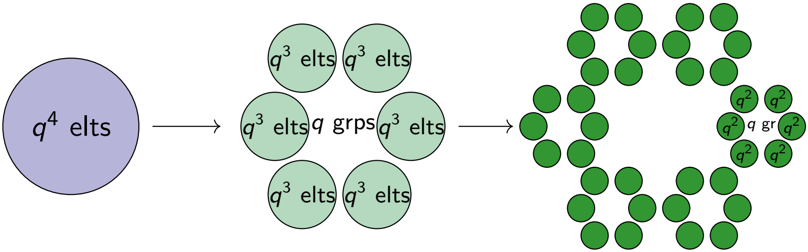

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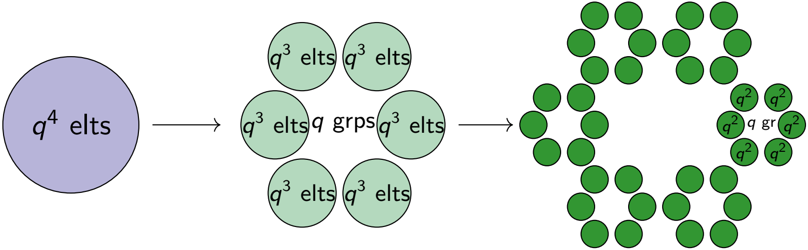

What is simple ? To consider that:

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2. Given (q^2), generate equations involving only poly in it and degree 1, 2 and 3 polynomials.

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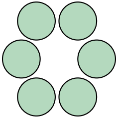

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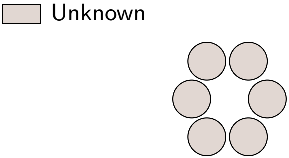

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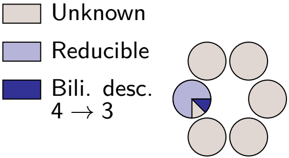

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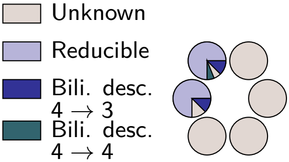

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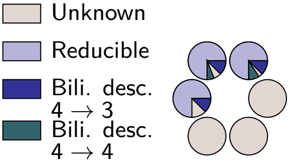

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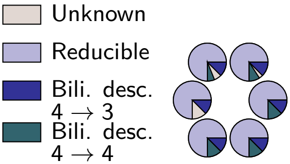

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Unknown

Reducible

Bili. desc. $4 \rightarrow 3$

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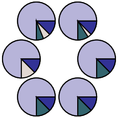
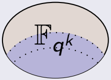


$\Rightarrow$ Final complexity of extension to deg 4 in $O(q^6)$ operations.

Main Result

Final asymptotic complexity of the three first phases:

$O(q^6)$ operations – to be compared with previous $O(q^7)$.



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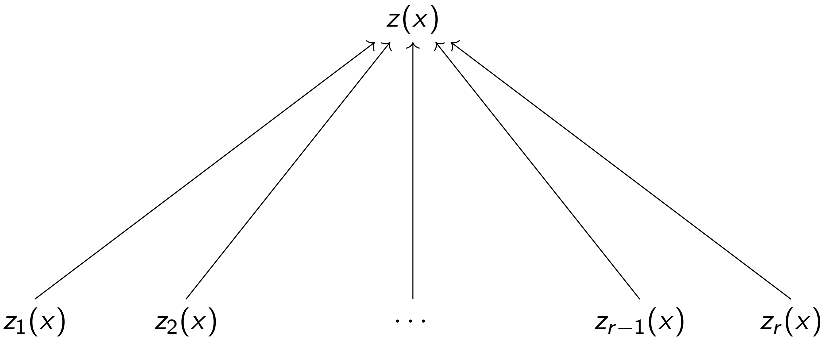
- ZigZag descent (all even degrees)

General principle

- Given target $z(x)$ in finite field, write:

$$z(x) = \prod_i z_i(x)^{e_i}, \quad \text{with smaller } z_i\text{s}$$



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$$F(x, x^{p^{\ell_1}})^{p^{\ell_2}} = F(x^{p^{\ell_2}}, h_0(x)/h_1(x)) \quad \text{in } \mathbb{F}_{q^k}.$$

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Modern descent strategies

- Remember basic Equation:

$$h_1(X)^D B(X) \prod_{\alpha \in \mathbb{F}_q} (A(X) - \alpha B(X)) = [A, B]_D(X)$$

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- Make $z(x)$ appear on the right or left
  - On the right: bilinear descent
  - On the left: quasi-polynomial
  - On the right (powers of two): ZigZag descent [GKZ14]

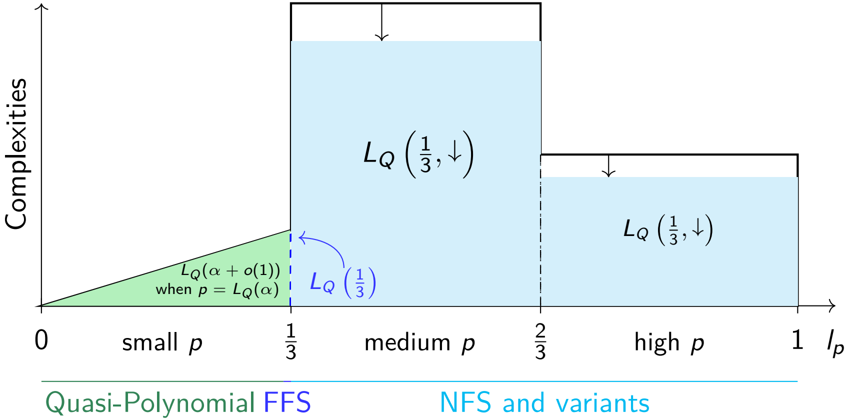
Descent Tree

- Continued fractions, **at most one application**
- Classical descent, **many levels possible**
- Bilinear descent (or [GKZ14]), **in practice 4-5 levels max.**
- Quasi-polynomial descent **in practice 2 levels max.**

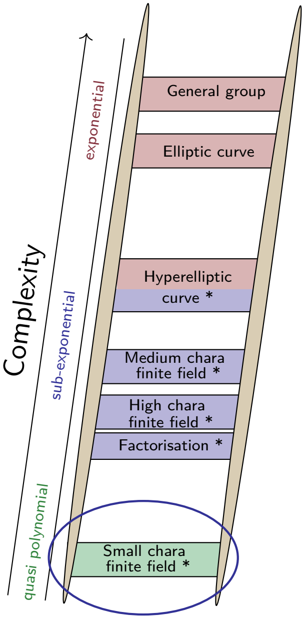
Practical application

- Record in characteristic 3 on $\mathbb{F}_{3^{5 \cdot 479}}$, a finite field of cardinality a 3796-bit integer.
  - Not a special extension field such as Kummer extension !
  - Make use of the Dual Frobenius Representation combined with the useful variant (both not presented here).
- To be compared with previous record in characteristic 3 by Adj, Menezes, Oliveira and Rodriguez-Henriquez on a 1551-bit finite field.
- Time : 8600 CPU-hours $\approx$ 1 CPU-year

Complexities of Index Calculus Algorithms



Questions ?


Complexity
exponential
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General group
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