Curve41417: Karatsuba revisited

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Joint work with Daniel J. Bernstein and Tanja Lange

Real-World Application

Performance budget

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  - $n, P \mapsto nP$
  - ARM Cortex-A8

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- Curve41417 (security level above 2^{200})
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Design of Curve41417

- High-security elliptic curve (security level above 2^{200})
- Defined over prime field $\mathbf{F}_p$ where $p = 2^{414} - 17$
- In Edwards curve form

$$x^2 + y^2 = 1 + 3617x^2y^2$$

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- Large prime-order subgroup (cofactor 8)
- IEEE P1363 criteria (large embedding degree, etc.)
- Twist secure, i.e., twist of Curve41417 also secure

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- Constant-time table-lookup:
  - read entire table
  - select via arithmetic
    if c is 1, select tbl[i]
    if c is 0, ignore tbl[i]

$$t = (t \cdot (1 - c)) + (\text{tbl[i]} \cdot (c))$$
$$t = (t \ \texttt{and} \ (c - 1)) \ \texttt{xor} \ (\text{tbl[i]} \ \texttt{and} \ (-c))$$

ECC Arithmetic

- Mix coordinate systems:
  - doubling: projective X, Y, Z
  - addition: extended X, Y, Z, T

  (See https://hyperelliptic.org/EFD/)

- Scalar multiplication:
  - signed fixed windows of width $w = 5$
  - precompute $0P, 1P, 2P, \ldots, 16P$
    also multiply $d = 3617$ to T coordinate
  - special first doubling
  - compute T only before addition

ARM Cortex-A8 Vector Unit

- 128-bit vector
- Arithmetic and load/store unit can perform in parallel
- Operate in parallel on vectors of four 32-bit integers or two 64-bit integers
- Each cycle produces:

  four 32-bit integer additions: $a_0+b_0, a_1+b_1, a_2+b_2, a_3+b_3$

  or

  two 64-bit integer additions: c_0+d_0, c_1+d_1

  or

  one multiply-add instruction: $a_0 b_0 + c_0$

  where a_i, b_i are 32- and c_i, d_i are 64-bit integers

Redundant Representation

- Use non-integer radix $2^{414/16} = 2^{25.875}$
- Decompose integer f modulo $2^{414} - 17$ into 16 integer pieces
- Write f as

$$
\begin{array}{llll}
f_0 + & 2^{26} f_1 + & 2^{52} f_2 + & 2^{78} f_3 + \\
2^{104} f_4 + & 2^{130} f_5 + & 2^{156} f_6 + & 2^{182} f_7 + \\
2^{207} f_8 + & 2^{233} f_9 + & 2^{259} f_{10} + & 2^{285} f_{11} + \\
2^{311} f_{12} + & 2^{337} f_{13} + & 2^{363} f_{14} + & 2^{389} f_{15}
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  e.g.: $3 \cdot (8 \times 8) \rightarrow 3 \cdot (3 \cdot (4 \times 4))$ + even more additions
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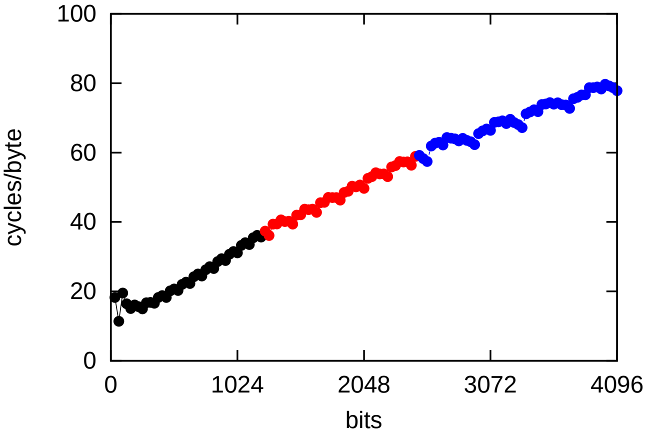
- What is the zero-level/one-level cutoff for number of limbs?

GMP's cutoffs for Karatsuba



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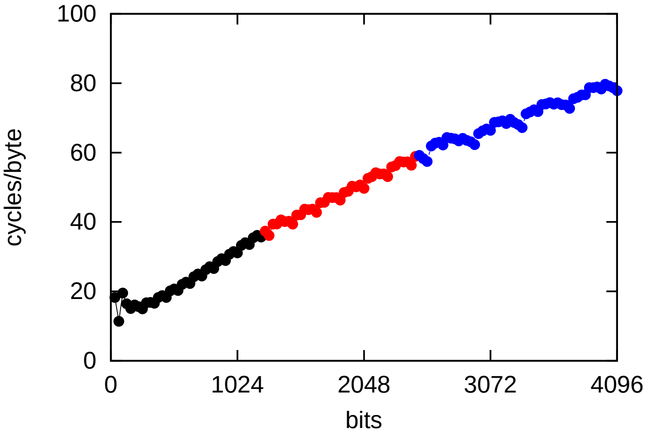

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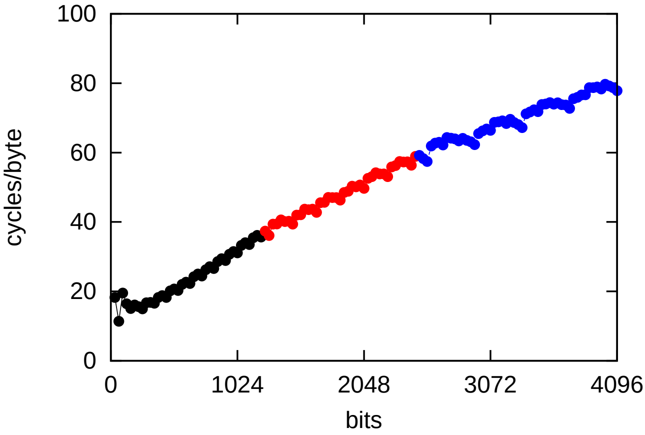

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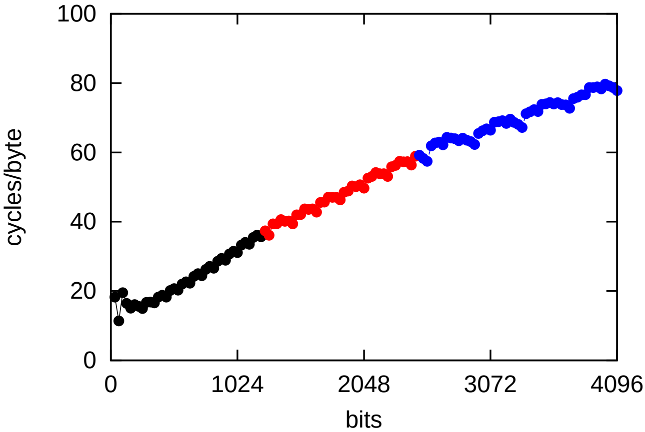

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- We reduce cutoff via redundant representation

Polynomial Multiplication

- Goal: Compute $P = AB$
  given $A = a_0 + a_1 t^n$ and $B = b_0 + b_1 t^n$
- Method 1: schoolbook
  $P = a_0 b_0 + (a_0 b_1 + a_1 b_0) t^n + a_1 b_1 t^{2n}$
- Method 2: Karatsuba ($8n-4$ additions)
  $P = a_0 b_0 + ((a_0+a_1)(b_0+b_1) - a_0 b_0 - a_1 b_1) t^n + a_1 b_1 t^{2n}$
- Method 3: refined Karatsuba ($7n-3$ additions)
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- Method 4: reduced refined Karatsuba ($6n-2$ additions) (new)
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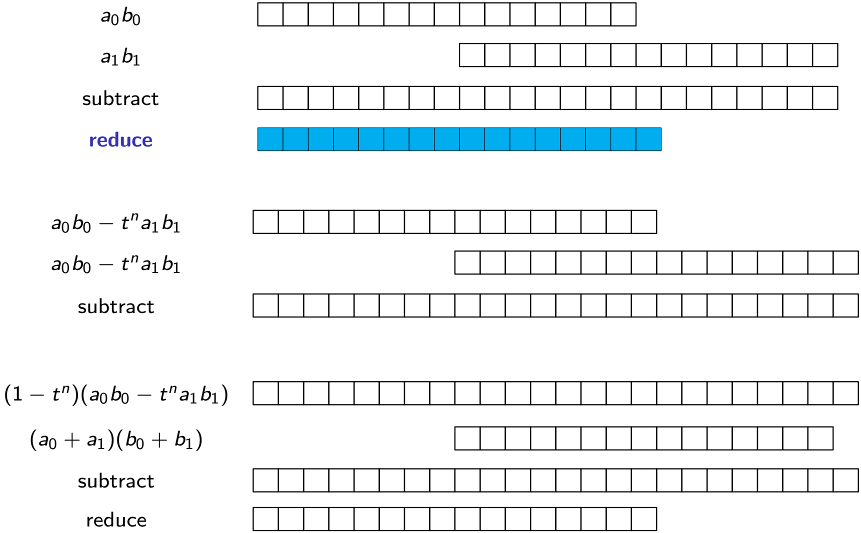
Reduced Refined Karatsuba



Cost Comparison (Karatsuba)

Level	Mult.	Add		Cost
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0-level	256	15	0	$256+\ 8+\ 0=264$
1-level	192	59	16	$192+30+\ 4=226$
2-level	144	119	40	$144+60+10=214$
3-level	108	191	76	$108+96+19=223$

Note: use multiply-add instructions

Recall:

1 cycle per multiplication

0.5 cycle per 64-bit addition

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2-level	144	103	40	$144+52+10 = 206$
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curve	# cycle on i.MX515	# cycle on Sitara
secp160r1	$\approx$ 2.1 million	$\approx$ 2.1 million
nistp192	$\approx$ 2.9 million	$\approx$ 2.8 million
nistp224	$\approx$ 4.0 million	$\approx$ 3.9 million
nistp256	$\approx$ 4.0 million	$\approx$ 3.9 million
nistp384	$\approx$ 13.3 million	$\approx$ 13.2 million
nistp521	$\approx$ 29.7 million	$\approx$ 29.7 million

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- Coming soon: Intel Haswell implementation with Sebastian Verschoor