Robust Coin Flipping

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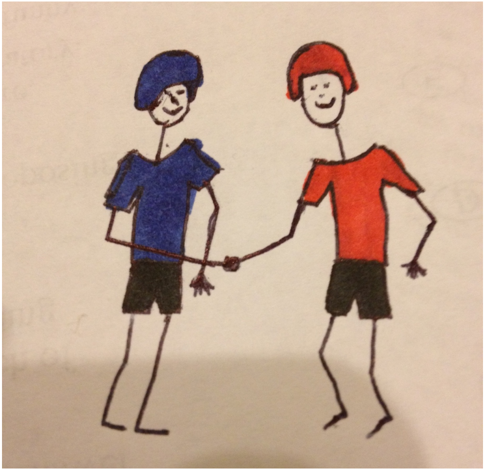
Sport Match



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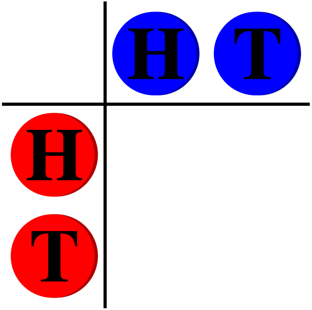

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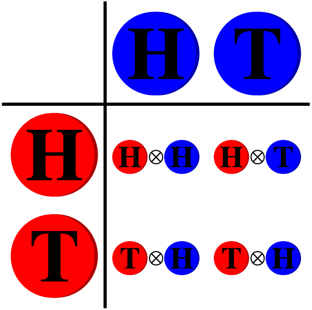

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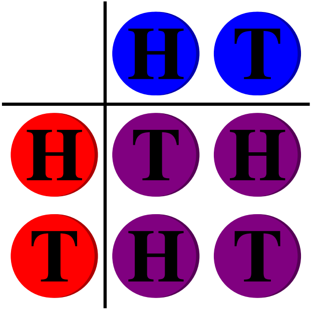

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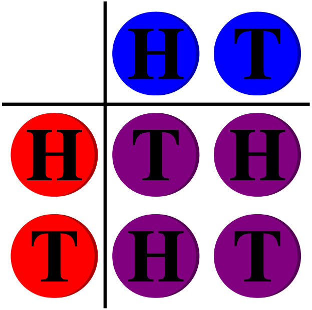

Probability of **H**?

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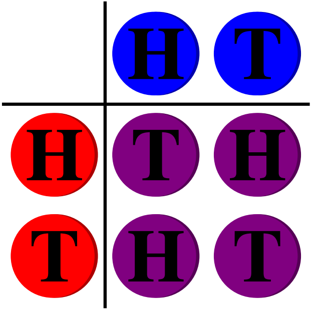

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?

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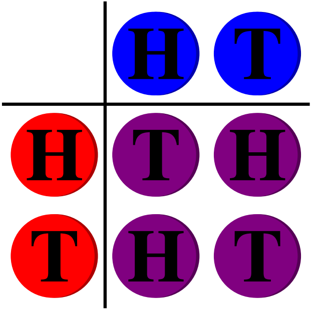

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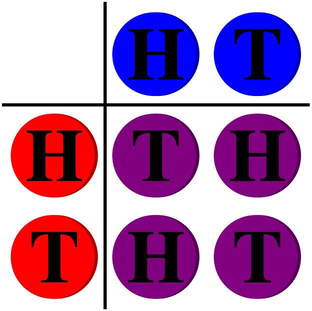

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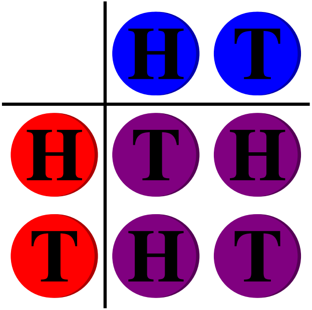

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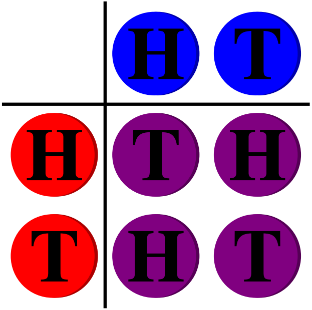
Sport Match



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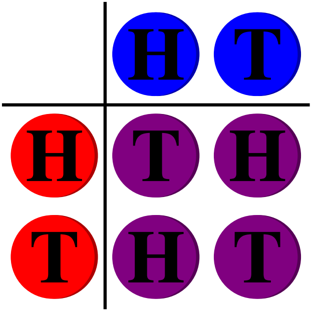

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Once you discover A the problem is easy.

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"Does there exist a multilinear operator with these properties? If so, can I construct one?"
3. Draw on a rich array of techniques in algebraic geometry to to find or disprove the key multilinear operator

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But...
We hope to convince you that
the techniques are *serious* and *practical*.

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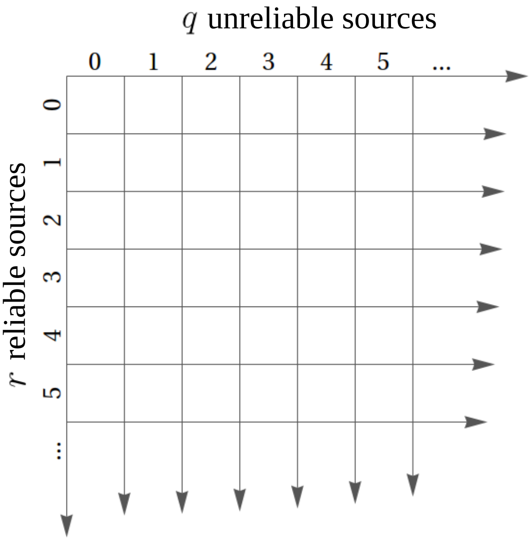
  the probability of tails is $1 - \alpha$.

Results



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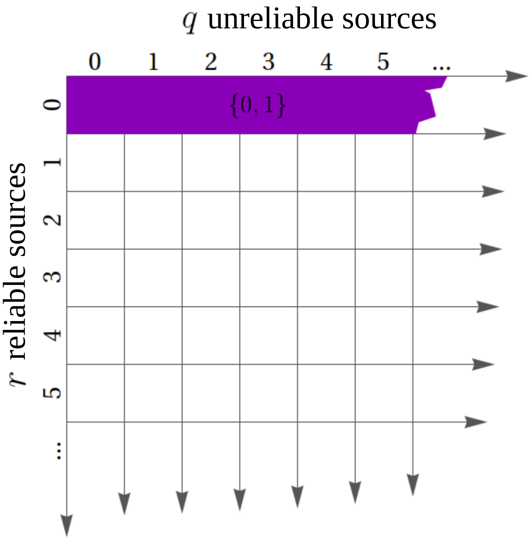

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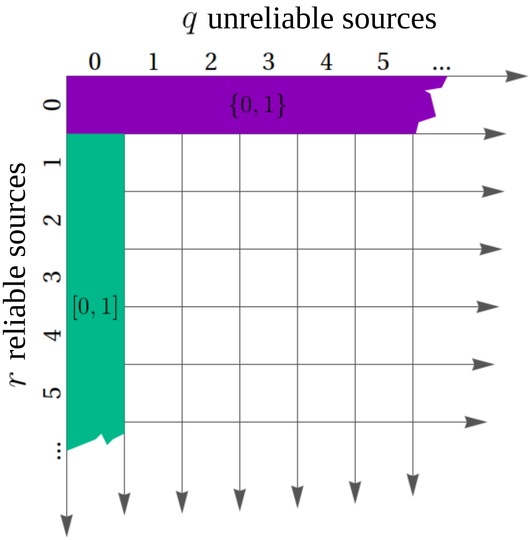

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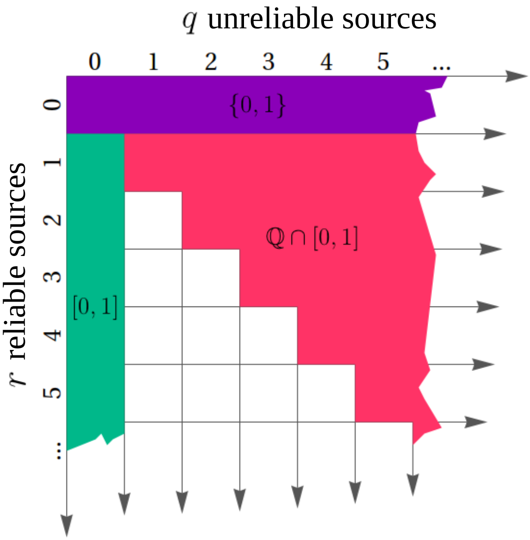

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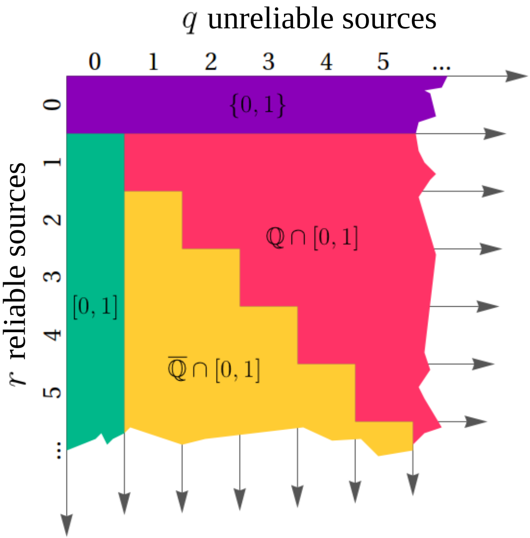

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Works if even one source is reliable (i.e. if $r \geq 1$)

When $p = 2$, Every α is Rational

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Any bilinear form A has at most one associated α.

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When $p = 2$, $\alpha \in \mathbb{Q}$.

- Since A is a zero-one matrix, it is fixed by any field automorphism of $\mathbb{C}/\mathbb{Q}$
- But any nontrivial Galois conjugate of α would violate the lemma!

Restatement using Multilinear Algebra

For $p = 3$, $q = 1$, we want to find a $\{0, 1\}$-hypermatrix A and probability vectors $\beta^{(i)}$ such that, for all probability vectors $x^{(i)}$,

$$\alpha = A(x^{(1)}, \beta^{(2)}, \beta^{(3)}) = A(\beta^{(1)}, x^{(2)}, \beta^{(3)}) = A(\beta^{(1)}, \beta^{(2)}, x^{(3)}).$$

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\alpha &= \frac{\sqrt{5}-1}{2}
\end{aligned}
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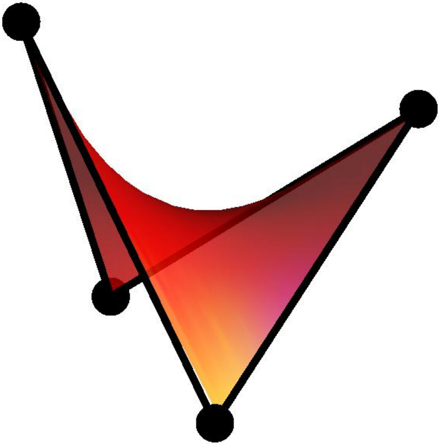
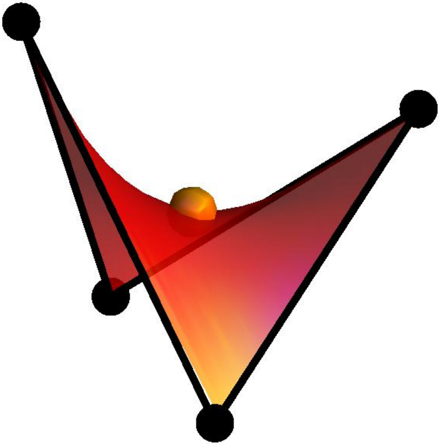
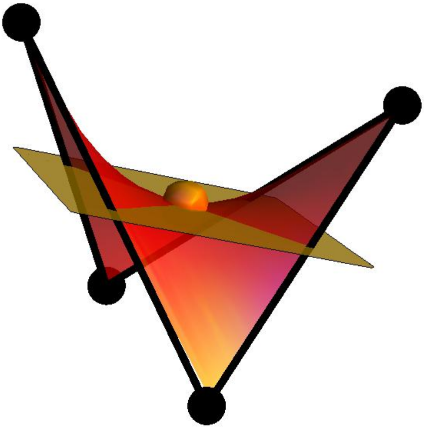
$$\iff \mathrm{Det}(\alpha J - A) = 0$$







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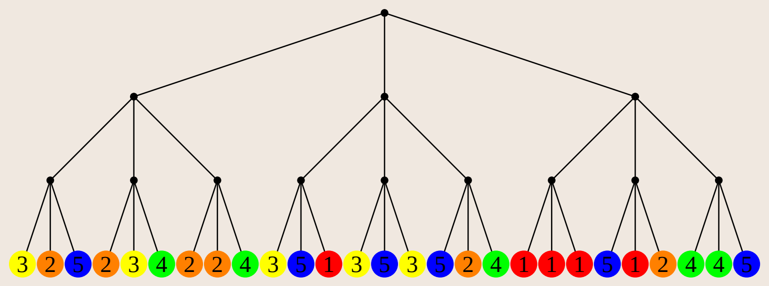
Proof in two steps:

- Use algebraic geometry to produce a point on the variety
- Use Diophantine approximation and analysis to wiggle the solution into the positive cone

Constructing any Algebraic α

- Deduce general case from $p = 3$, $q = 1$ case using the Bureaucracy Lemma.



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- Thank you!