Hard-to-Compute Bits for Elliptic Curve-Based One-Way Functions

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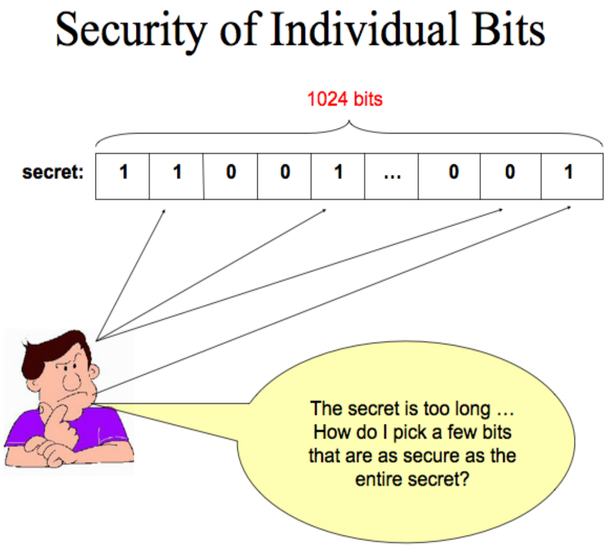
Security of Individual Bits



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Our Contribution

Assuming the hardness of FAPI-2, we show that all the bits of the input to the pairing-based one-way function are secure.

(Short) Weierstrass Equations

Equations $E_{a,b} : y^2 = x^3 + ax + b,\ a, b \in \mathbb{F}_p,\ 4a^3 + 27b^2 \neq 0.$

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Isomorphism classes

Two elliptic curves $E_{a,b}$ and $E_{a',b'}$ are isomorphic (over $\mathbb{F}_p$) **if and only if** $a' = \lambda^{-4}a,\ b' = \lambda^{-6}b$ for some $\lambda \in \mathbb{F}_p^\times$. The isomorphism between $E_{a,b}$ and $E_{a',b'}$ is given by

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Each isomorphism class thus contains precisely $p - 1$ short Weierstrass equations.

The main result

All bits of the pairing-based OWF are hard-to-compute

If there is an oracle that predicts the kth bit of the input to f_Q on a significant fraction of all short Weierstrass equations in an isomorphism class then there is an efficient algorithm to invert f_Q.

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Conclusion

Thus, if FAPI-2 is hard, all the bits of the input of the pairing-based OWF are hard-to-compute.

The result is in fact much more general as few properties of the pairing-based function f_Q are used.

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Bit Security for EC-based OWFs

Let $\mathbb{G}$ be an elliptic curve group and $f\colon \mathbb{G} \to \mathbb{G}_T$ be any function with the property that its definition is independent of the choice of short Weierstrass equation in the isomorphism class (e.g., the pairing-based OWF). Assuming that inverting f is hard, every bit of the input to f is secure.

Elliptic Curve-Based OWFs

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Open Question: Are there other cryptographically interesting EC-based OWFs besides the pairing-based functions for which this result could apply?

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  - Heavy Fourier coefficients: computation of heavy Fourier coefficients (a version of the SFT algorithm by Akavia–Goldwasser–Safra)
  - Recoverability: for a given frequency, find all inputs having large Fourier coefficient at this frequency (a technique of Morillo–Ràfols).

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Define a noisy codeword $w\colon \mathbb{F}_p^\times \to \{\pm 1\}$ as follows

$$w(\lambda) := \mathcal{B}(W_\lambda, f_Q(R)),$$

where $W_\lambda\colon y^2 = x^3 + \lambda^{-4}ax + \lambda^{-6}b$.

To each point R and each short Weierstrass equation W, one can associate a function (codeword) $C_R^W : \mathbb{F}_p^\times \to \{\pm 1\}$

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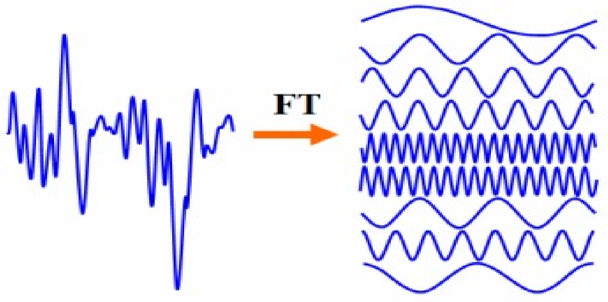

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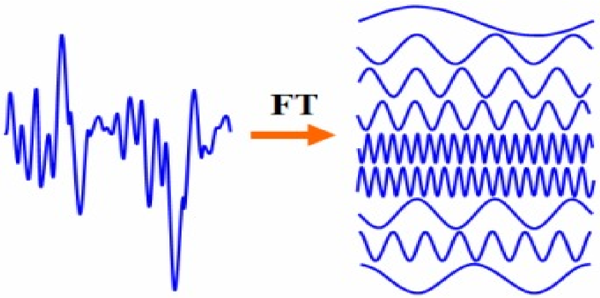

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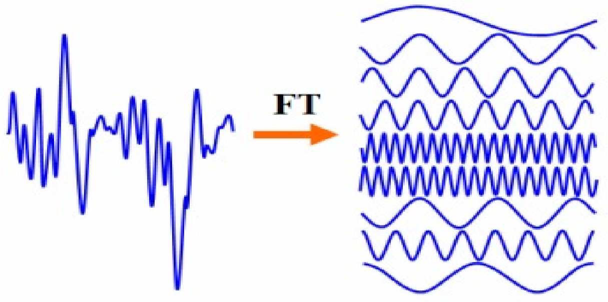

- Fourier basis formed out of different frequencies (in our case, characters),
- A function is Fourier concentrated if the number of significant frequencies (characters) is small.

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Given a frequency, find (in polynomial time) all codewords for which this frequency (character) is significant (i.e., has a large Fourier coefficient).

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Fourier concentration + Recoverability $\Rightarrow$ List Decoding

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One needs a different list-decoding problem!

Using an idea of Boneh–Shparlinski:

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- Define a new prediction oracle as follows:

$$\mathcal{B}'(W_\lambda, f_Q(R)) = \begin{cases} \mathcal{B}(W_{r(\lambda)}, f_Q(R)), & \text{if } \lambda \in \mathbb{F}_p^2 \\ \text{most probable value of } B_k(x) & \text{else,} \end{cases}$$

  where $r\colon \mathbb{F}_p^2 \to \mathbb{F}_p$ is a random square root function.

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