Pseudorandom Functions and Lattices

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CRYPTO 2011 Rump Session
16 Aug 2011

Pseudorandom Functions

- A family $\mathcal{F} = \{F : \{0,1\}^k \to \{0,1\}\}$ s.t. $(F \leftarrow \mathcal{F}) \overset{c}{\approx}$ rand function.

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   - ✔ Low-depth: NC^2, NC^1 or even TC^0
   - ✗ Huge circuits that need mucho preprocessing
   - ✗ No "post-quantum" construction under standard assumptions

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2. Main technique: "derandomization" of LWE: deterministic errors

“Learning With Rounding” (LWR)

Learning With Errors (LWE)   [Regev’05]

- Distinguish pairs $(\mathbf{a}_i\ ,\ b_i = \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i) \in \mathbb{Z}_q^n \times \mathbb{Z}_q$ from uniform
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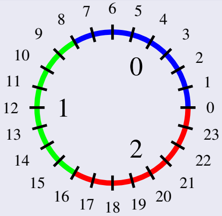
- Let $q > p$ and define $\lfloor x \rceil_p = \lfloor (p/q) \cdot x \rceil \bmod p$.

  (Common in decryption to "remove noise.")



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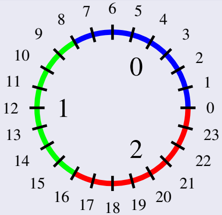

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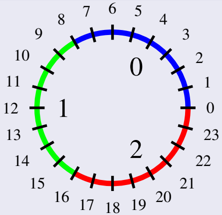

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- We prove LWE $\leq$ LWR for appropriate parameters

Synthesizer-Based PRF (*a la* [NR'95])

Synthesizer from LWR

- For random $\mathbf{a}_1, \mathbf{a}_2, \ldots$ and $\mathbf{s}_1, \mathbf{s}_2, \ldots$ (mod q),

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PRF on Domain $\{0,1\}^{k=2^d}$ (e.g. $d = 7$)

- (Public) moduli $q_d > q_{d-1} > \cdots > q_0$.
- Secret key is $2k$ square matrices $\mathbf{S}_{i,b} \in \mathbb{Z}_{q_d}^{n \times n}$ for $i \in [k]$, $b \in \{0,1\}$.

(Or ring elements $s_{i,b}$ using ring-LWR [LPR'10].)

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- Depth $d = \lg k$ tree of LWR synthesizers:

$$F(x_1 \cdots x_8) =$$

$$\left\lfloor \left\lfloor \lfloor \mathbf{S}_{1,x_1} \cdot \mathbf{S}_{2,x_2} \rceil_{q_2} \cdot \lfloor \mathbf{S}_{3,x_3} \cdot \mathbf{S}_{4,x_4} \rceil_{q_2} \right\rceil_{q_1} \cdot \left\lfloor \lfloor \mathbf{S}_{5,x_5} \cdot \mathbf{S}_{6,x_6} \rceil_{q_2} \cdot \lfloor \mathbf{S}_{7,x_7} \cdot \mathbf{S}_{8,x_8} \rceil_{q_2} \right\rceil_{q_1} \right\rceil_{q_0}$$

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Details: ePrint report #2011/401