Recovering Short Generators of Principal Ideals in Cyclotomic Rings

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Principal ideals in cryptography

Let $\mathbb{K}$ be a numberfield (e.g. $= \mathbb{Q}(\zeta_m)$) and R its ring of integer ($R = \mathbb{Z}[\zeta_m]$).

A few cryptosystems, for example:

- Soliloquy [Campbell et al., 2014]
- FHE [Smart and Vercauteren, 2010]
- Graded encoding schemes [Garg et al., 2013, Langlois et al., 2014]

share this Key Generation procedure.

KeyGen

sk Choose a "short" $g \in R$ as a private key

pk Give a bad $\mathbb{Z}$-basis $\mathbf{B}$ of the ideal (g) as a public key (e.g. HNF).

Short generator recovery

Cryptanalysis in two steps (Key Recovery Attack)

1. Principal Ideal Problem (PIP)
   - Given a $\mathbb{Z}$-basis $\mathbf{B}$ of a principal ideal $\mathfrak{I}$,
   - Recover some generator h (i.e. $\mathfrak{I} = (h)$)

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2. Short Generator Problem
   - Given an arbitrary generator $h \in R$ of $\mathfrak{I}$
   - Recover g (or some g' equivalently short)

Cost of those two steps

1. Principal Ideal Problem (**PIP**)
   - sub-exponential time ($2^{\tilde{O}(n^{2/3})}$) classical algorithm [Biasse and Fieker, 2014, Biasse, 2014].
   - quantum polynomial time algorithm [Eisenträger et al., 2014, Campbell et al., 2014, Biasse and Song, 2015].
2. Short Generator Problem
   - equivalent to the **CVP** in the log-unit lattice
   - becomes a **BDD** problem in the crypto cases.
   - claimed to be easy [Campbell et al., 2014] for the m^{th}-cyclotomic ring when $m = 2^k$
   - confirmed by experiments [Schank, 2015]

This Work

We focus on step 2, and prove it can be solved in classical polynomial time for the aforementioned cryptanalytic instances, when the ring R is the ring of integers of the cyclotomic number field $K = \mathbb{Q}(\zeta_m)$ for $m = p^k$.

The Problem

Short generator recovery

Given $h \in R$, find a small generator g of the ideal (h).

Note that $g \in (h)$ is a generator iff $g = u \cdot h$ for some unit $u \in \mathbb{R}^\times$.
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Translation an to additive problem

Take logarithms:

$$\mathrm{Log} : g \mapsto (\log|\sigma_1(g)|, \ldots, \log|\sigma_n(g)|) \in \mathbb{R}^n$$

where the σ_i's are the canonical embeddings $\mathbb{K} \to \mathbb{C}$.

The Unit Group and the log-unit lattice

Let $R^\times$ denotes the multiplicative group of units of R. Let

$$\Lambda = \operatorname{Log} R^\times.$$

Theorem (Dirichlet unit Theorem)

$\Lambda \subset \mathbb{R}^n$ *is a lattice (of a given rank).*

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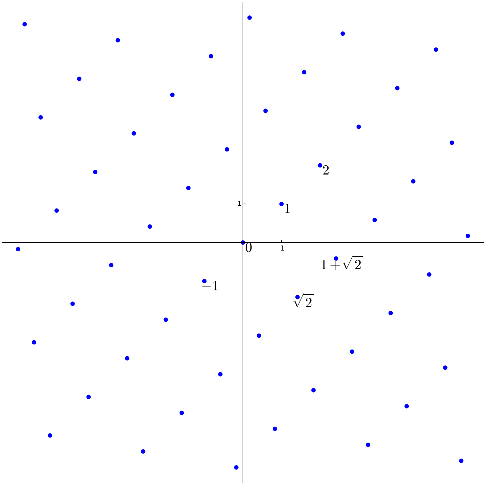
Reduction to a Close Vector Problem

Elements g is a generator of (h) if and only if

$$\operatorname{Log} g \in \operatorname{Log} h + \Lambda.$$

Moreover the map Log preserves some geometric information:
g is the "smallest" generator iff $\operatorname{Log} g$ is the "smallest" in $\operatorname{Log} h + \Lambda$.

Example: Embedding $\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{R}^2$



- x-axis: $\sigma_1(a + b\sqrt{2}) = a + b\sqrt{2}$
- y-axis: $\sigma_2(a + b\sqrt{2}) = a - b\sqrt{2}$
- component-wise additions and multiplications

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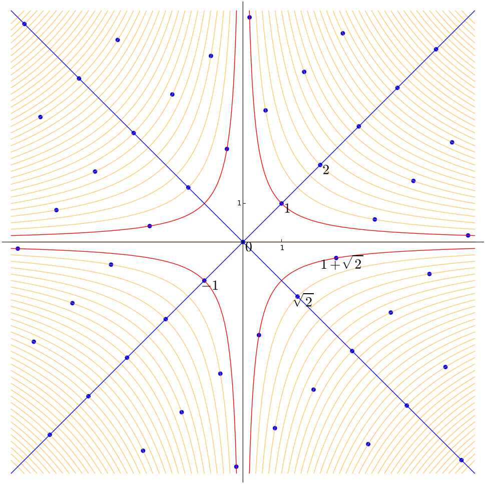

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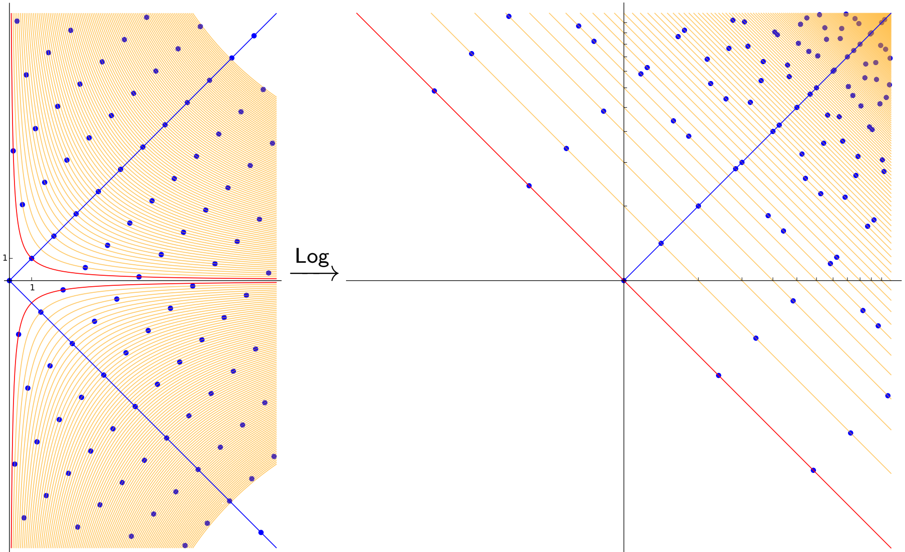
Blue: "Orthogonal" elements
Red: Units (algebraic norm 1)
Orange: "Isonorms" curves

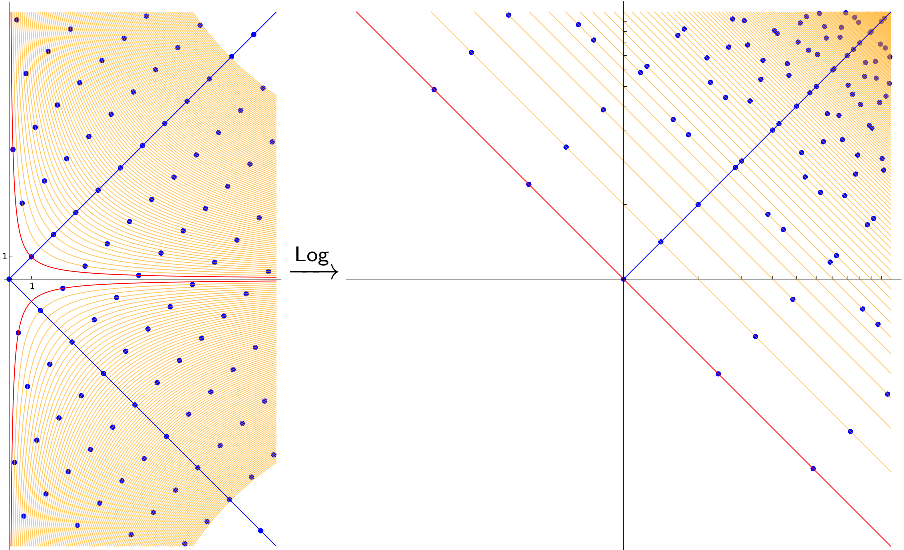
Example: Logarithmic Embedding Log $\mathbb{Z}[\sqrt{2}]$

$(\{\bullet\}, +)$ is a sub-monoid of $\mathbb{R}^2$



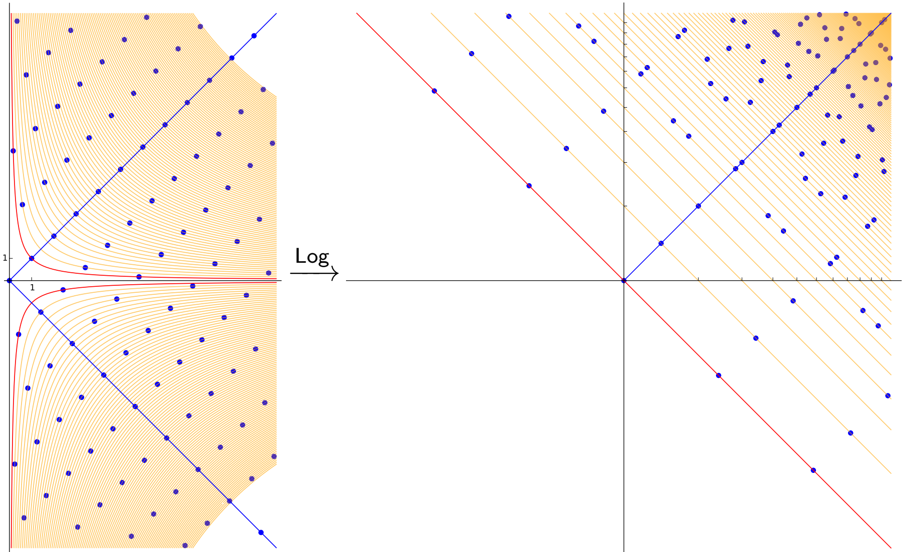
Example: Logarithmic Embedding $\operatorname{Log} \mathbb{Z}[\sqrt{2}]$

$\Lambda = (\{\bullet\}, +) \cap \diagdown$ is a lattice of $\mathbb{R}^2$, orthogonal to $(1, 1)$



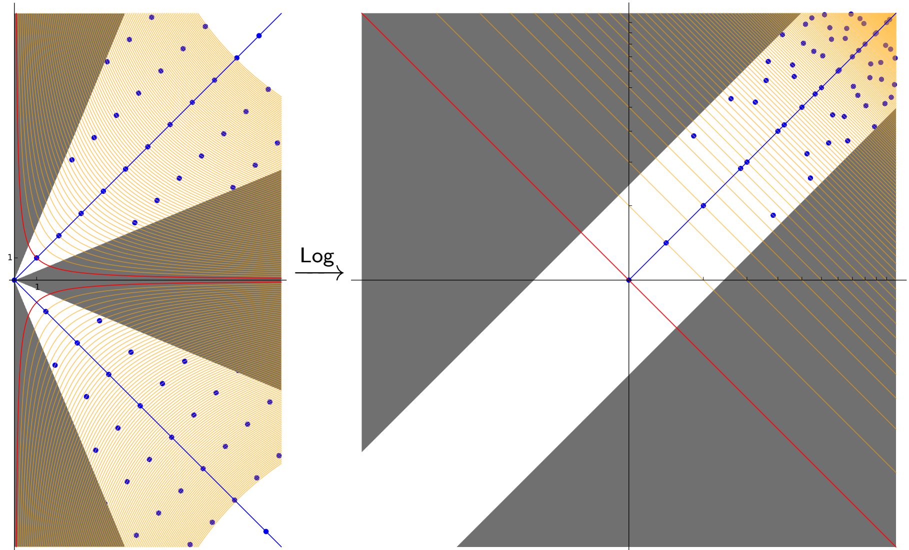
Example: Logarithmic Embedding $\mathrm{Log}\,\mathbb{Z}[\sqrt{2}]$

$\{\bullet\} \cap \diagdown$ are shifted finite copies of Λ



Reduction modulo $\Lambda = \operatorname{Log} \mathbb{Z}[\sqrt{2}]^\times$

The reduction $\bmod \Lambda$ for various fundamental domains.



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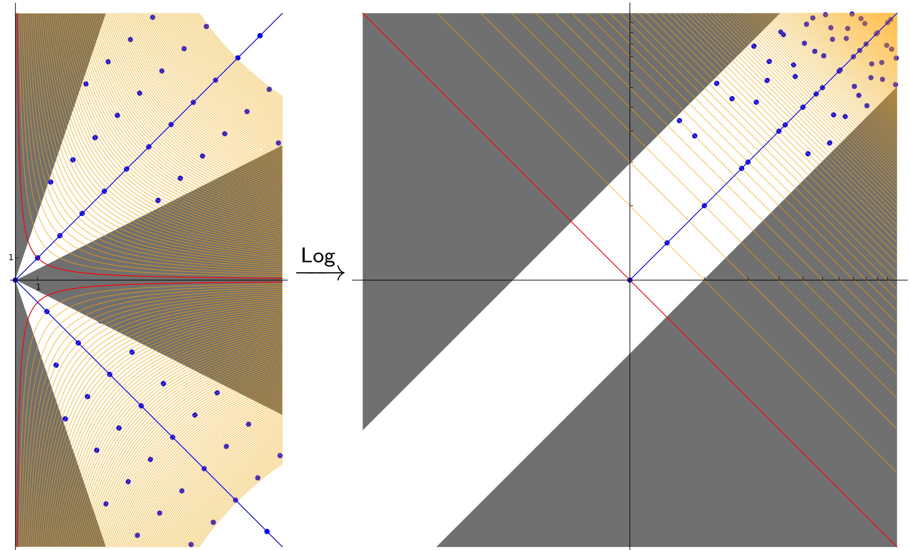

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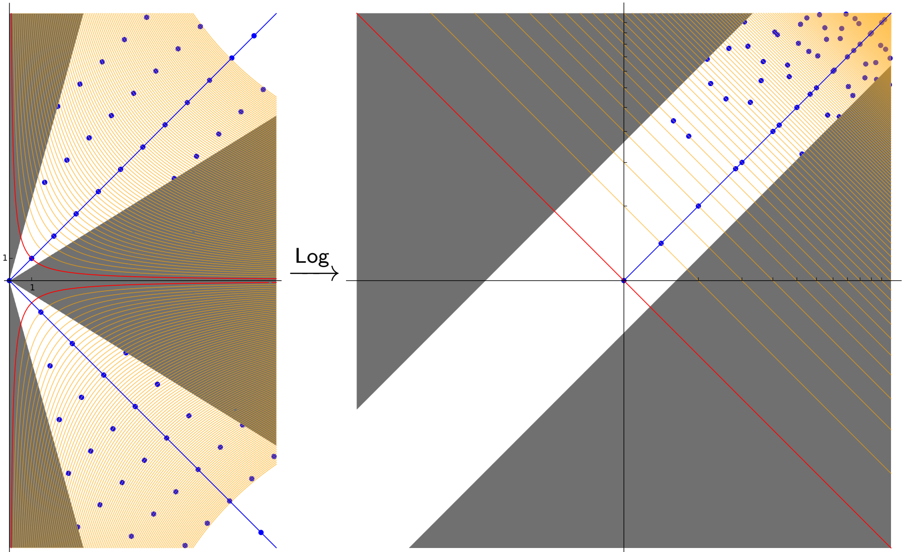

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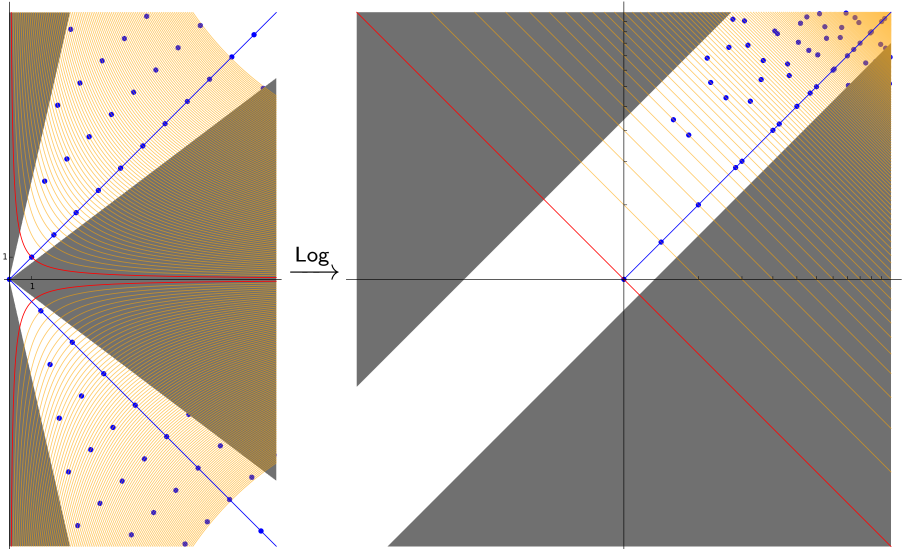

Round-Off Decoding

We also need the fundamental domain to have an efficient reduction algorithm. The simplest one follows:

$\textsc{Round}(\mathbf{B}, \mathbf{t})$ for $\mathbf{B}$ a basis of Λ

- Return $\mathbf{B} \cdot \mathrm{frac}(\mathbf{B}^{-1} \cdot \mathbf{t})$.

Used as a decoding algorithm, its correctness is characterized by the error $\mathbf{e}$ and the *dual basis* $\mathbf{B}^\vee = \mathbf{B}^{-T}$.

Fact [Lenstra, 1982, Babai, 1986]

Suppose $\mathbf{t} = \mathbf{v} + \mathbf{e}$ for some $\mathbf{v} \in \Lambda$. If $\langle \mathbf{b}_j^\vee, \mathbf{e} \rangle \in [-\frac{1}{2}, \frac{1}{2})$ for all j, then

$$\textsc{Round}(\mathbf{B}, \mathbf{t}) = \mathbf{v}.$$

Recovering Short Generator: Proof Plan

Folklore strategy [Bernstein, 2014, Campbell et al., 2014] to recover a short generator g

1. Construct a basis $\mathbf{B}$ of the unit-log lattice $\operatorname{Log} R^{\times}$
   - For $K = \mathbb{Q}(\zeta_m)$, $m = p^k$, an (almost[1]) canonical basis is given by

$$\mathbf{b}_j = \operatorname{Log} \frac{1-\zeta^j}{1-\zeta}, \quad j \in \{2, \dots, m/2\}, j \text{ co-prime with } m$$

2. Prove that the basis is "good", that is $\|\mathbf{b}_j^{\vee}\|$ are all small
3. Prove that $\mathbf{e} = \operatorname{Log} g$ is small enough

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Technical contributions

2. Estimate $\|\mathbf{b}_j^\vee\|$ precisely using analytic tools [Washington, 1997, Landau, 1927]
3. Bound $\mathbf{e}$ using theory of sub-exponential random variables [Vershynin, 2012]

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Geometric statement from Analytic Number Theory

Theorem ([Landau, 1927])

If χ is a non-quadratic Dirichlet character of conductor f.

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Theorem (Cramer, D. , Peikert, Regev)

Let $m = p^k$, and $\mathbf{B} = (\mathrm{Log}(b_j))_{j \in G \setminus \{1\}}$ be the canonical basis of $\mathrm{Log}\, C$. Then, for all j

$$\left\| \mathbf{b}_j^\vee \right\|^2 \leq O\left(m^{-1} \cdot \log^3 m\right).$$

Interpretation

The log-unit lattice $\mathrm{Log}\, R^\times$ admits a (known, efficiently computable) basis that is **almost orthogonal**: **BDD** is easy !

No Crypto from Principal Ideals

We formalized, generalized and proved a claim of [Campbell et al., 2014]:

Corollary [Cramer, D. , Peikert, Regev] (simplified)

If g follows a reasonable distribution, then given any generator h of (g), one may recover g in poly-time with probability $1 - o(1)$.

Combined with a poly-time quantum algorithm[2] of [Biasse and Song, 2015], this breaks several cryptographic proposal.

[2] Alt. a classical sub-exponential algorithm [Biasse and Fieker, 2014, Biasse, 2014].

What about the worst case ?

Theorem [Cramer, D. , Peikert, Regev]

Given a generator h of any principal ideal (h), one may find in poly-time a generator g of (h) of length

$$\|g\| \leq N(h)^{1/n} \cdot 2^{\tilde{O}(\sqrt{n})}.$$

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We also show that this is nearly optimal:

Theorem [Cramer, D. , Peikert, Regev]

In some principal ideals $\mathfrak{I}$, the shortest generator has length at least

$$\|g\| \geq N(\mathfrak{I})^{1/n} \cdot 2^{\Omega(\sqrt{m}/\log m)}.$$

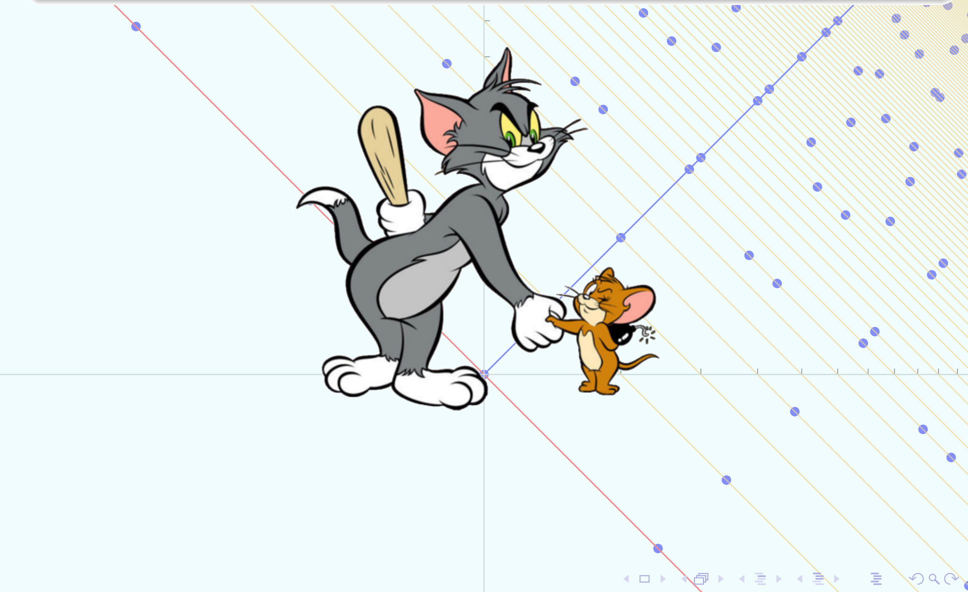
Open questions

1. Are there other classes of rings whose log-unit lattice can be studied ?
   - For cyclotomics, several happy event for the proof to go through.
   - Other rings are harder to study. Security by ignorance ?

2. Does this result has a bearing on (worst-case) non-principal ideals ?
   - Possibly: class group Caley graphs, Stickleberger's Ideal . . .
   - This approach seems limited to large approx. factors $2^{\tilde{O}(\sqrt{n})}$.

3. And on Ring-LWE ?
   - Seems much harder than 2 .
   - Would still be limited to large approx. factors $2^{\tilde{O}(\sqrt{n})}$.

Questions ?



Questions ?

Thanks for your attention !

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