Multi-property-preserving Domain Extension Using Polynomial-based Modes of Operation

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Merkle-Damgård transform

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- The most popular way to build a cryptographic hash function from a fixed-size compression function
- Preserves collision resistance with an appropriate padding algorithm

If computing collisions becomes somehow feasible for the underlying compression function, then the hash function may fail worse than expected

Generic attacks

- Multicollision attack (Joux, Eurocrypt 2004)
- Long-message second preimage attack (Kelsey and Schneier, Eurocrypt 2005)
- Herding attack (Kelsey and Kohno, Eurocrypt 2006), etc.

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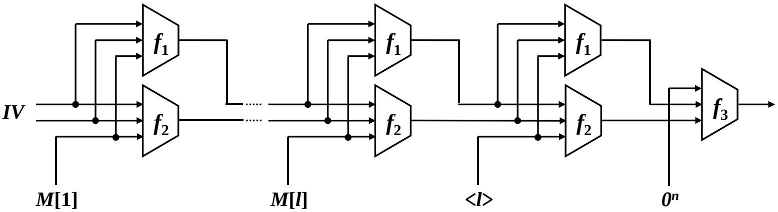
Wide-pipe strategy

Double-piped mode of operation

- The aforementioned weaknesses can be mitigated by increasing the size of the internal state (Lucks, Asiacrypt 2005)
- The internal state size should be seen as a security parameter of its own right
- Lucks proposed to use a “narrow" compression function in a double-piped mode



Security of double-piped mode of operation

Yasuda analyzed the security of the double-piped mode of operation as a multi-property-preserving domain extension (Eurocrypt 2009)

As a secure message authentication code (MAC)

Preserves unforgeability up to $O(2^{5n/6})$ query complexity

As a pseudorandom function

Preserves indistinguishability up to $O(2^n)$ query complexity

As a pseudorandom oracle

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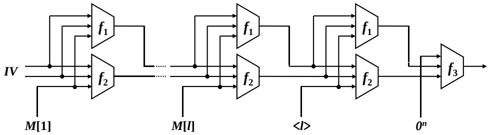
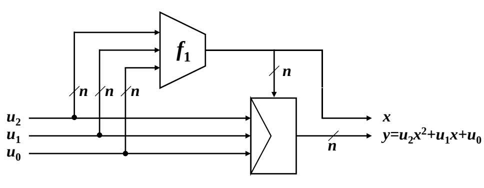


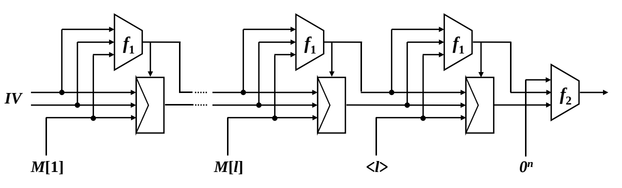
Polynomial-based mode of operation

Efficiency(rate) can be improved by replacing the second primitive by a polynomial

Compression function $\phi[f_1]$



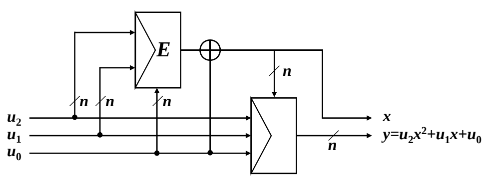
Hash function $H[f_1, f_2]$



Refinements

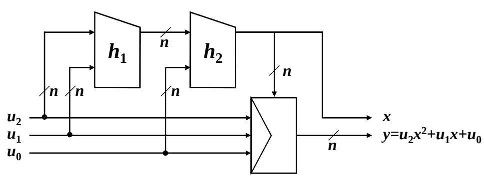
With f_1 replaced by a $2n$-bit key blockcipher in DM-mode:

Quadratic BC-based function



With f_1 replaced by the cascade of two $2n$-n bit primitives:

Quadratic cascade function



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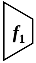
Modular approach for constructing MACs

An and Bellere presented a modular approach for domain extension of a FIL-MAC (Crypto 1999)

Construction of VIL-MAC

1. Construct a FIL-WCR using a FIL-MAC.
2. Using MD-transform, build a VIL-WCR.
3. The composition of the VIL-WCR and a FIL-MAC with an independent key yields a secure VIL-MAC.

The modular approach allows us to focus on the proof of WCR for the polynomial-based compression function



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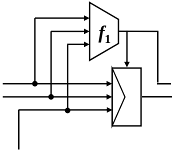

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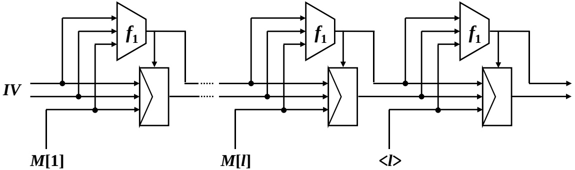

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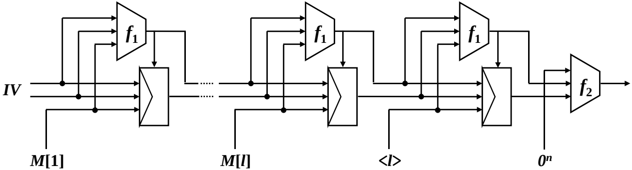

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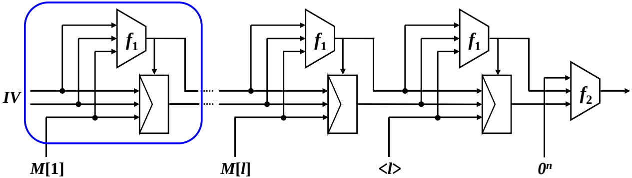


Security definitions

Given a function family f : Keys $\times$ Dom $\longrightarrow$ Rng,

Unforgeability

Experiment $\mathbf{Exp}_{\mathcal{A}}^{\mathrm{mac}}$

$k \stackrel{\$}{\leftarrow}$ Keys
$(m, \tau) \leftarrow \mathcal{A}^{f_k(\cdot)}$
if $f_k(m) = \tau$, m is “new" **then**
output 1
else
output 0

Weak collision resistance

Experiment $\mathbf{Exp}_{\mathcal{A}}^{\mathrm{wcr}}$

$k \stackrel{\$}{\leftarrow}$ Keys
$(m, m') \leftarrow \mathcal{A}^{f_k(\cdot)}$
if $f_k(m) = f_k(m')$,$m \neq m'$ **then**
output 1
else
output 0

- $\mathbf{Adv}_f^{\mathrm{mac}}(\mathcal{A}) = \mathbf{Pr}\left[\mathbf{Exp}_{\mathcal{A}}^{\mathrm{mac}} = 1\right]$
- $\mathbf{Adv}_f^{\mathrm{wcr}}(\mathcal{A}) = \mathbf{Pr}\left[\mathbf{Exp}_{\mathcal{A}}^{\mathrm{wcr}} = 1\right]$

Security results

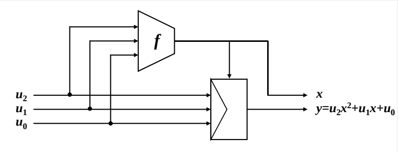
WCR of a polynomial-based compression function

Let $\phi[f]$ be the polynomial-based compression function defined by a function family $f : \{0,1\}^\kappa \times \{0,1\}^{3n} \to \{0,1\}^n$. Then,

$$\mathbf{Adv}^{\mathrm{wcr}}_{\phi[f]}(t,q) \le 2q(2+\log q)\mathbf{Adv}^{\mathrm{mac}}_{f}\left(t + O(n^2 q^4), q\right).$$

Proof

Compression function $\phi[f]$



Let $\mathcal{A}$ be an optimal adversary s.t.
$\mathbf{Adv}^{\mathrm{wcr}}_{\phi[f]}(\mathcal{A}) = \mathbf{Adv}^{\mathrm{wcr}}_{\phi[f]}(t, q) = \epsilon$

Interaction btwn $\mathcal{A}$ and f

- Oracle access to either $\phi[f]$ or f is equivalent
- $\mathcal{A}$'s query determines a quadratic curve
- f's response specifies a point on the curve

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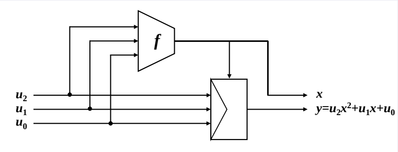

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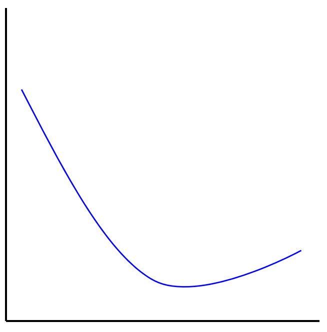

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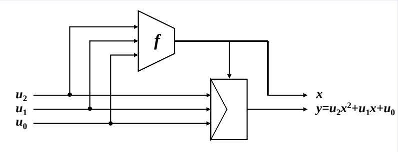

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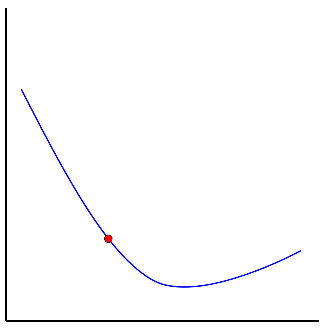

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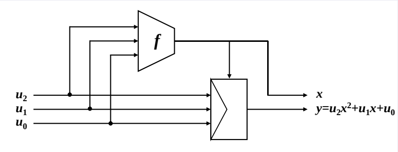

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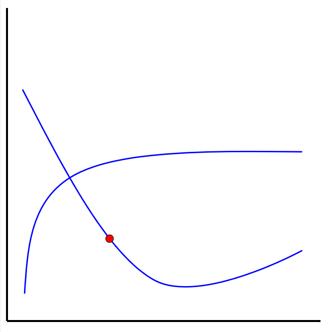

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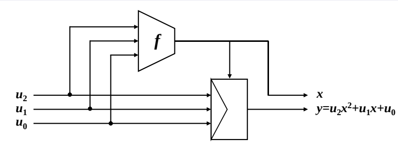

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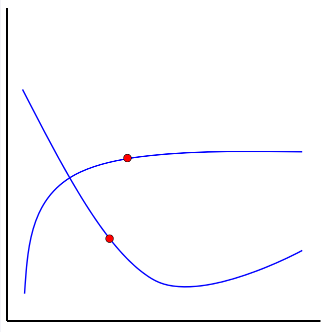

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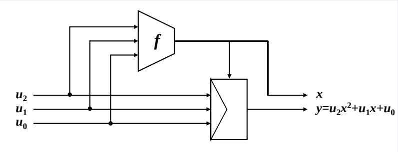

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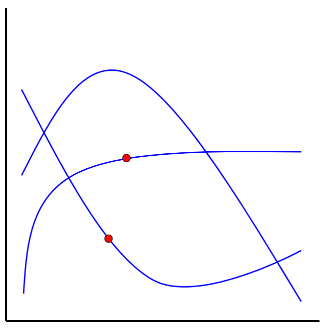

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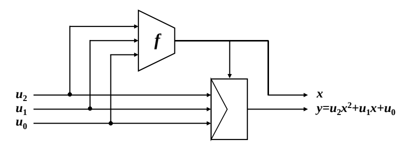

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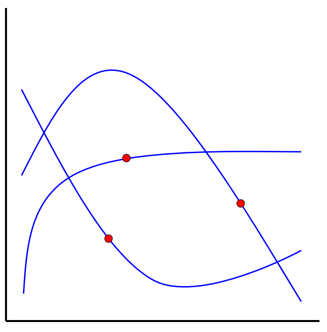

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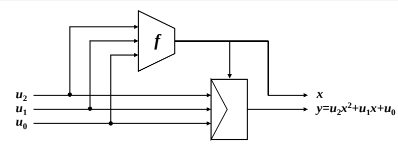

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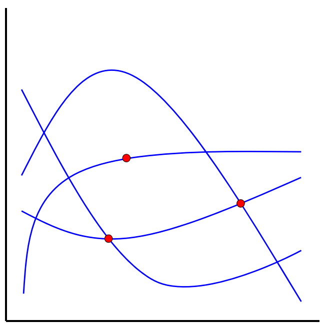

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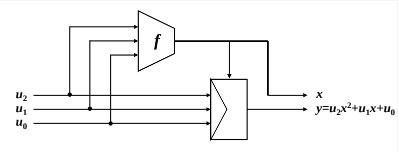

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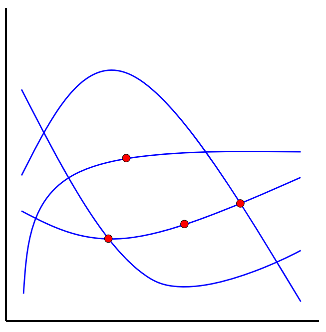

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Case analysis

- We will construct a forger $\mathcal{B}$ of f using $\mathcal{A}$ as a subroutine
- Let $\gamma = \max \gamma_i$, where $\gamma_i = \sharp$ points already placed on the i-th curve

Case 1

$\mathcal{A}$ finds a collision and $\gamma \leq \log q + 2$

Case 2

$\gamma > \log q + 2$

One of the two cases happens with probability at least $\epsilon/2$

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Case 1: $\mathbf{Pr}[\mathcal{A}$ finds a collision $\wedge \gamma \leq \log q + 2] \geq \epsilon/2$

Forger $\mathcal{B}$

1. $\mathcal{B}$ chooses $i \in \{1, \ldots, q\}$ uniformly at random
2. $\mathcal{B}$ runs $\mathcal{A}$ as a subroutine and faithfully answers the queries made by $\mathcal{A}$ until the $(i-1)$-th query
3. On the i-query, $\mathcal{B}$ presents a forgery by randomly choosing one of the points already placed on the i-th curve

Analysis

- With probability $\geq \frac{\epsilon}{2q}$, $\mathcal{B}$ makes a correct guess of the query that determines a collision
- In this case, $\mathcal{B}$ successfully forge f with probability $\geq \frac{1}{\log q+2}$
- Therefore, we have $\mathbf{Adv}_f^{\mathrm{mac}}(\mathcal{B}) \geq \frac{\epsilon}{2q(\log q+2)}$

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Case 2: $\mathbf{Pr}[\gamma > \log q + 2] \geq \epsilon/2$

Balls-in-bins game (Dodis and Steinberger, Crypto 2009)

Players: $\mathcal{A}$ and $\mathcal{B}$

Parameters: q, m_1, m_2, M ($m_1 < m_2$)

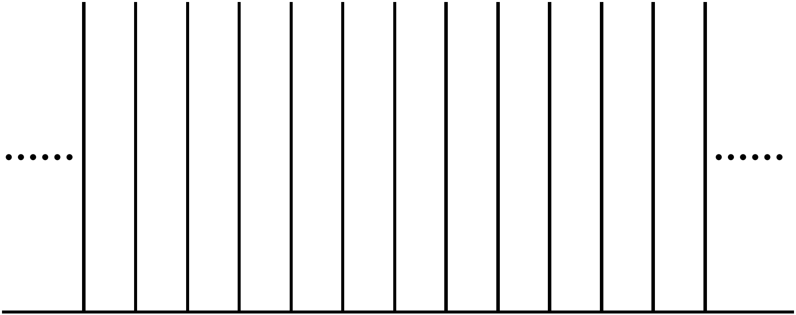
1. The game consists of q rounds
2. At each round, $\mathcal{A}$ publicly places a set of balls into a set of bins such that
    1. balls placed at the same round go into distinct bins,
    2. the number of bins containing more than m_1 balls at the end of the game is at most M,
    3. some bin eventually contains more than m_2 balls
3. Before each round, $\mathcal{B}$ can secretly “pass" or “guess" a bin that will receive a ball in the next round. $\mathcal{B}$ makes exactly one guess throughout the game
4. If $\mathcal{B}$ makes a correct guess, then $\mathcal{B}$ wins. Otherwise, $\mathcal{B}$ loses

Example: balls-in-bins game

Parameters

$q = 9,\ m_1 = 3,\ m_2 = 7,\ M = 5$

Round 1:



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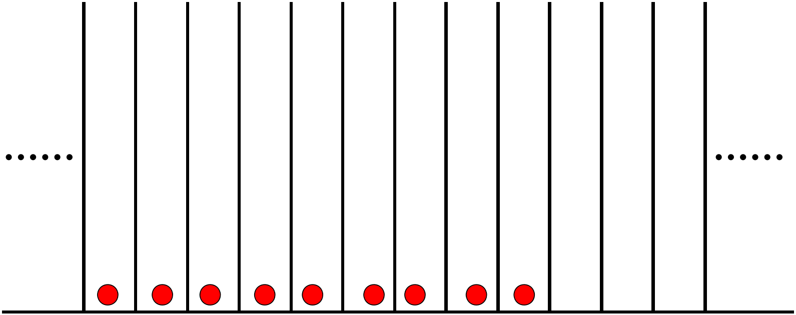

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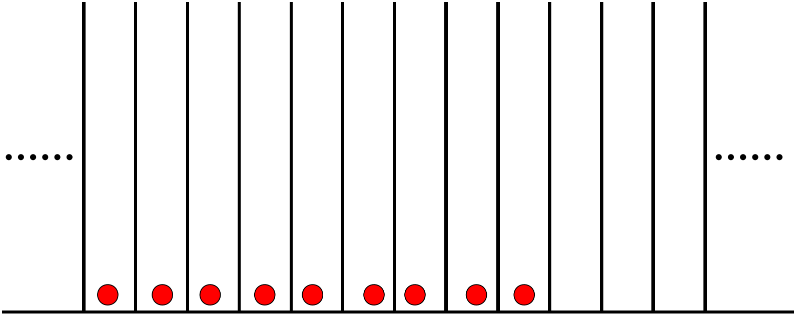

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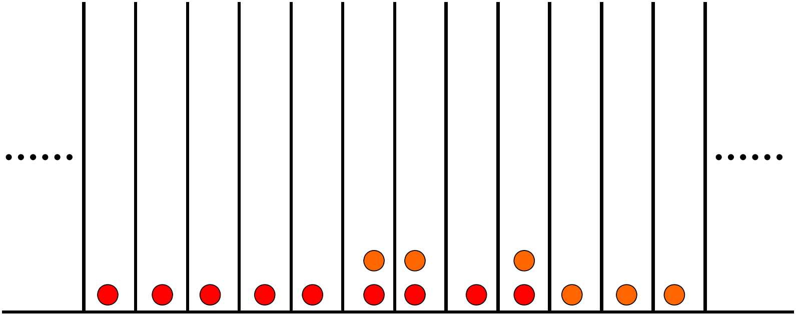

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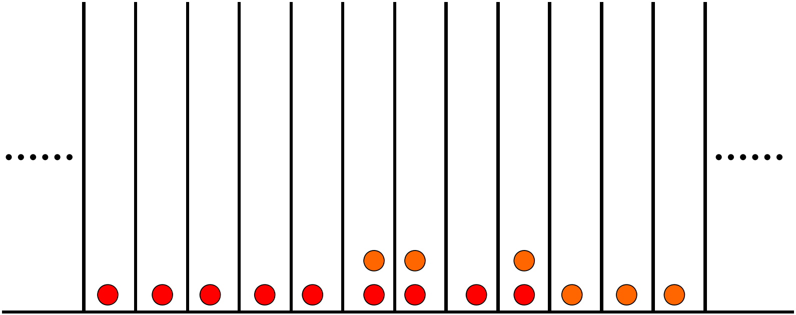

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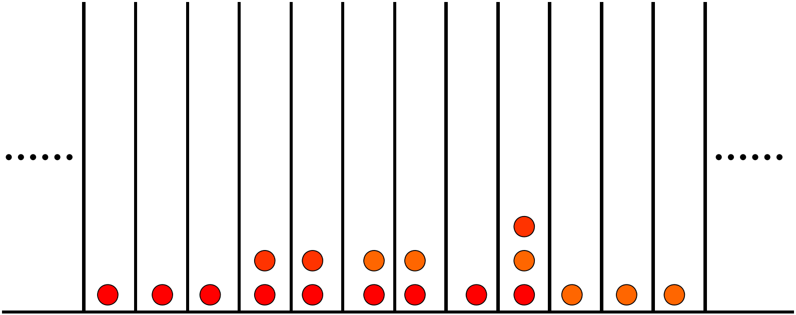
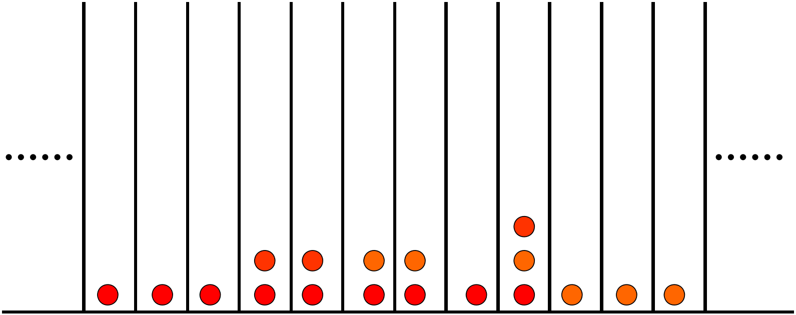


Example: balls-in-bins game

Parameters

$q = 9$, $m_1 = 3$, $m_2 = 7$, $M = 5$

Round 4:



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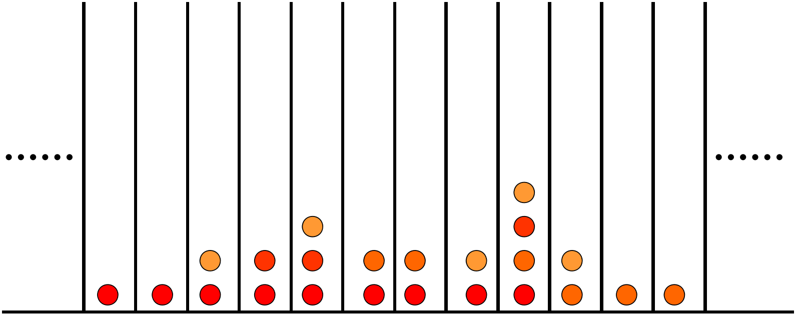

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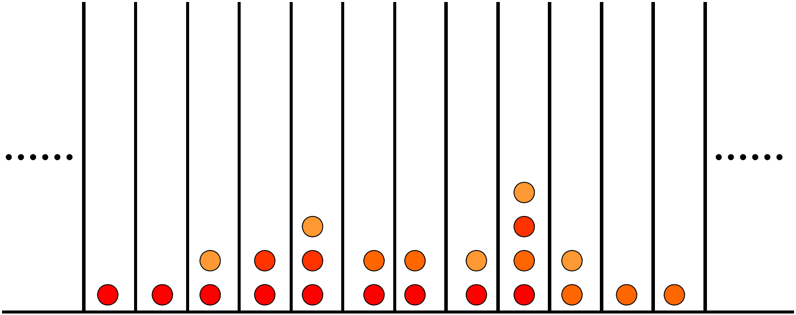
Round 5:



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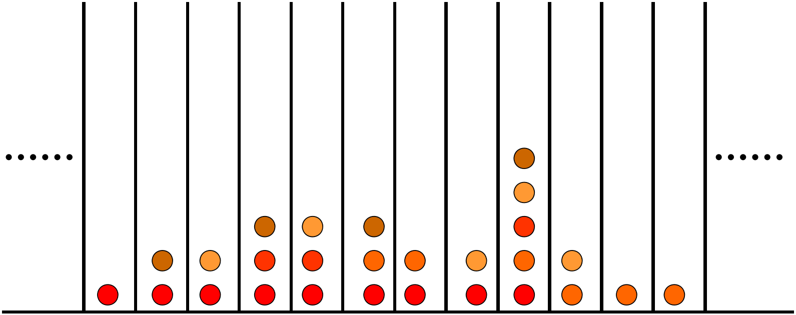

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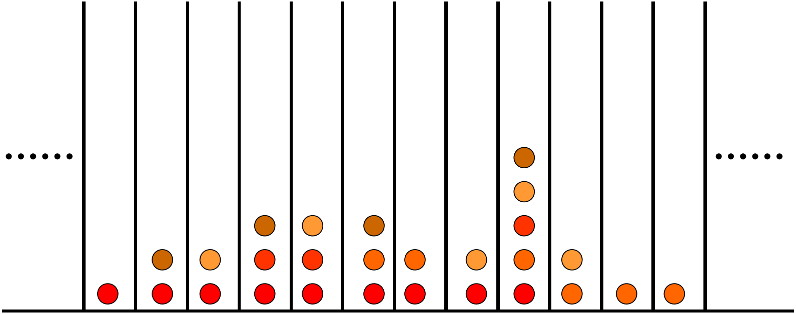
Round 6:



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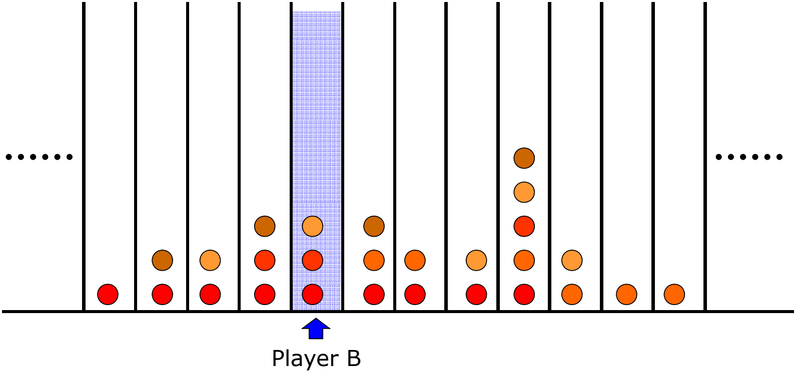

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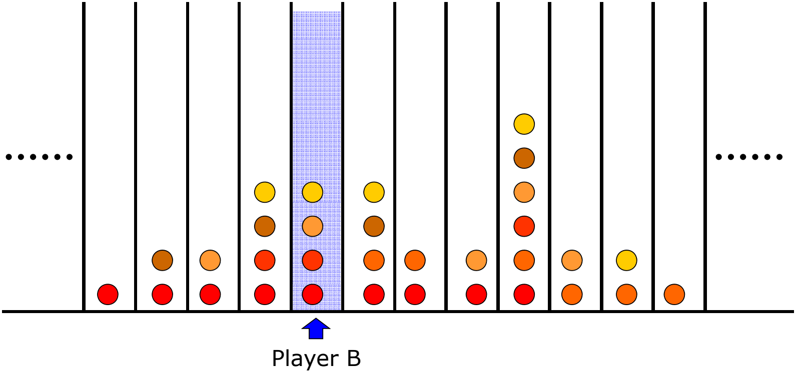

Example: balls-in-bins game

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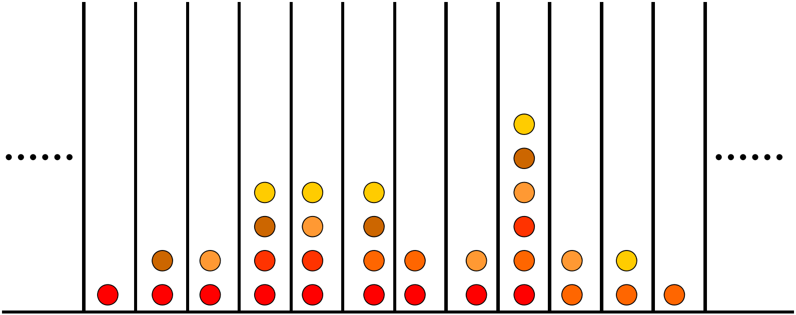
Round 7:



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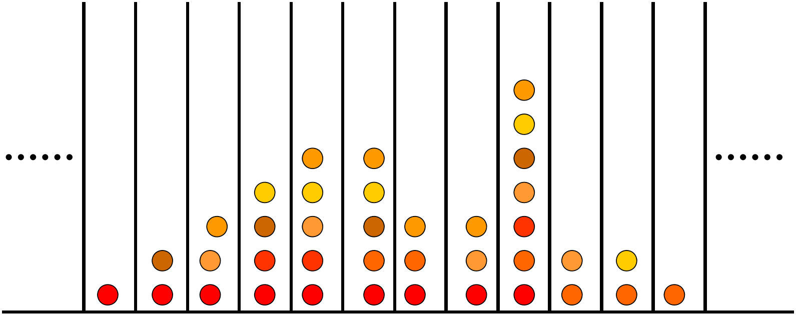

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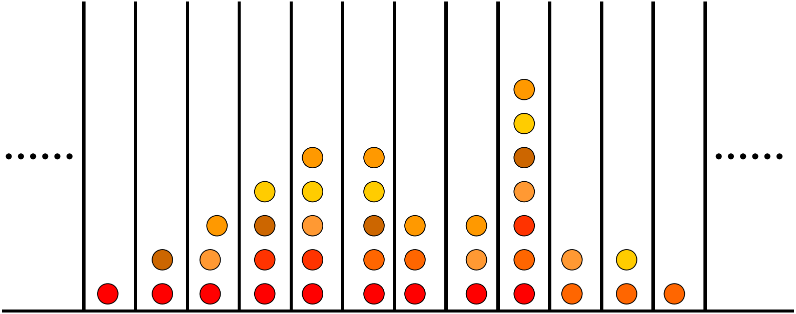
Round 8:



Example: balls-in-bins game

Parameters

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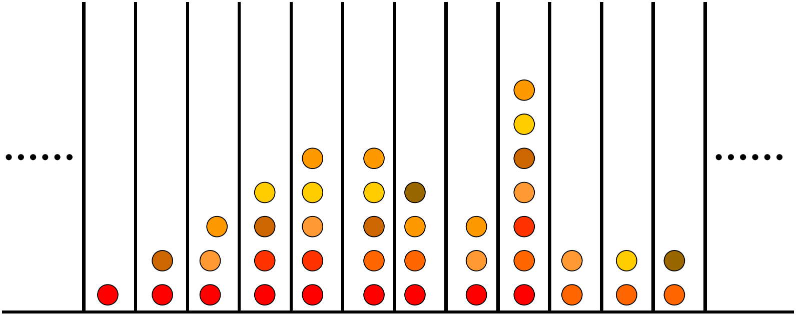

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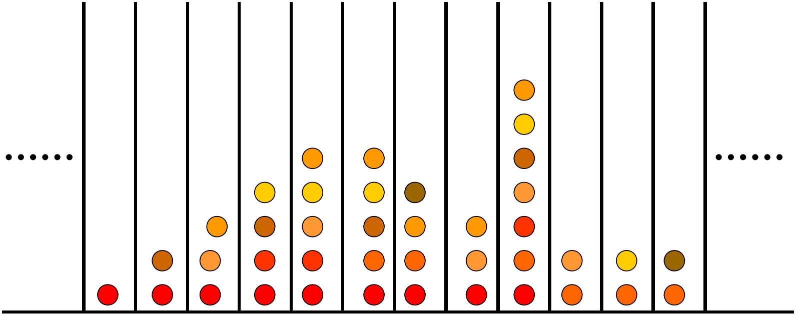

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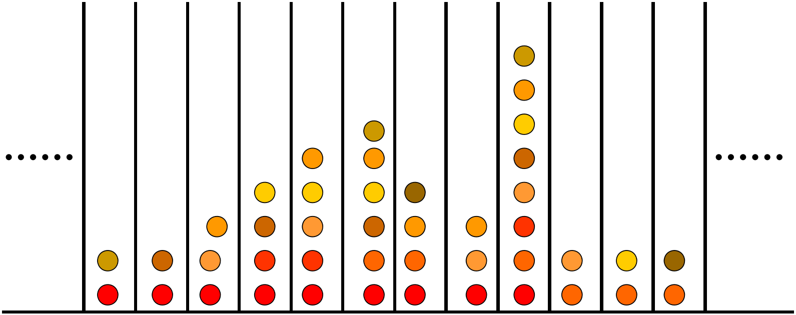

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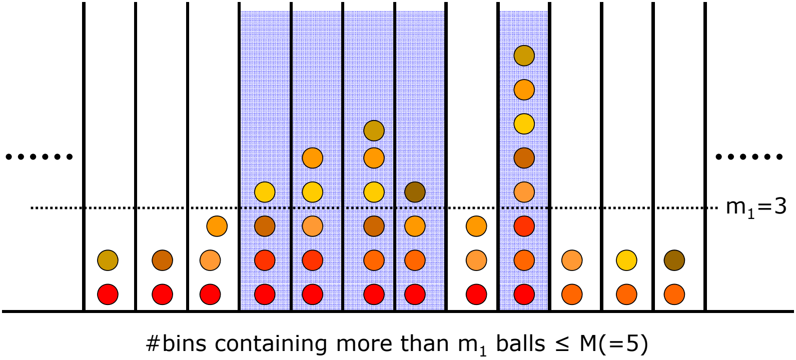
End of the game:



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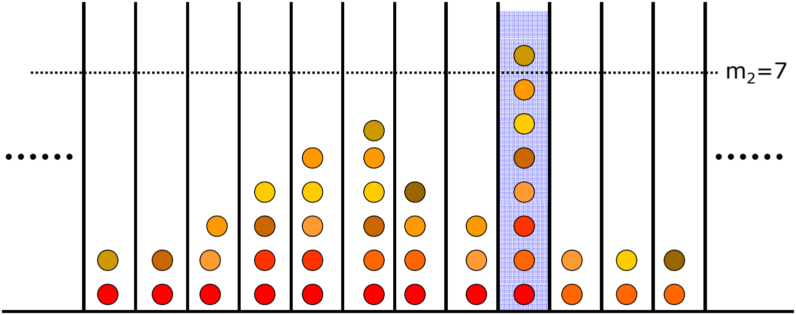

WCF adversary $\Rightarrow$ player $\mathcal{A}$ of a balls-in-bins game

balls-in-bins game: G

- A ball is associated with a point in $\mathbb{F}_{2^n}^2$
- A bin is a quadratic curve in $\mathbb{F}_{2^n}^2$
- On the i-th query $u[i]$ of $\mathcal{A}$, the i-th round of the game begins
- Given the i-th point $\phi[f](u[i])$ as a response of f, every quadratic curve containing the point except the i-th curve itself receives a single ball

Parameters

- The game consists of q rounds
- The number of bins that contain more than $m_1 = 2$ balls at the end of the balls-in-bins game is at most $\binom{q}{3} \leq M = q^3$
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A player $\mathcal{B}$ of game G with a high probability of winning can be transformed into a successful forger of f

If $\mathcal{B}$ makes a correct guess of the curve before the i-th round, then it can present a forgery of f by computing the intersection of the curve and the i-th curve (Probability $1/2$)

Winning strategy of player $\mathcal{B}$

Irrespective of $\mathcal{A}$'s strategy, there exists a strategy for $\mathcal{B}$ to win game G with probability at least $1/q \cdot 1/(qM)^{1/(m_2-m_1)}$

With $m_1 = 2$, $M = q^3$ and $m_2 = \log q + 2$, the player $\mathcal{B}$ can be transformed into a forger such that

$$\mathbf{Adv}_f^{\mathrm{mac}}(\mathcal{B}) \geq \frac{\epsilon}{2} \cdot \frac{1}{2} \cdot \frac{1}{q} \left(\frac{1}{qM}\right)^{1/(m_2-m_1)} = \frac{\epsilon}{64q}$$

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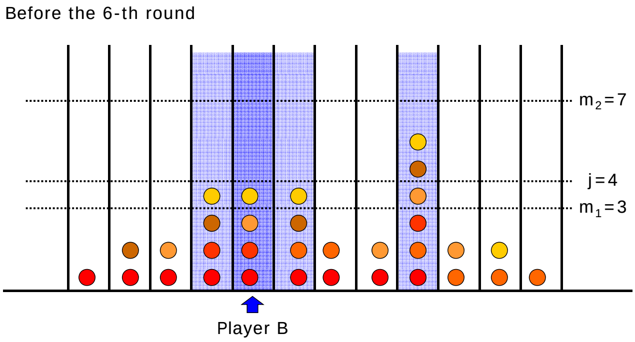
What is the winning strategy of $\mathcal{B}$?

$\mathcal{B}$'s strategy

1. Choose a round $i \in \{1, \ldots, q\}$ uniformly at random
2. Choose a level $j \in \{m_1 + 1, \ldots, m_2\}$ uniformly at random
3. Before the i-th round of the game, guess a bin uniformly at random from all bins containing at least j balls already



Summary

Case 1: $\mathbf{Pr}\left[\mathcal{A} \text{ finds a collision} \wedge \gamma \leq \log q + 2\right] \geq \epsilon/2$

There exists a forger $\mathcal{B}_1$ such that $\mathbf{Adv}_f^{\mathrm{mac}}(\mathcal{B}_1) \geq \frac{\epsilon}{2q(\log q+2)}$

Case 2: $\mathbf{Pr}\left[\gamma > \log q + 2\right] \geq \epsilon/2$

There exists a forger $\mathcal{B}_2$ such that $\mathbf{Adv}_f^{\mathrm{mac}}(\mathcal{B}_2) \geq \frac{\epsilon}{64q}$

For an optimal forger $\mathcal{B}$,

$$\begin{aligned}\mathbf{Adv}_f^{\mathrm{mac}}(\mathcal{B}) &\geq \min\left\{\frac{1}{2q(\log q+2)}, \frac{1}{64q}\right\} \times \epsilon \\ &= \frac{1}{2q(\log q+2)}\mathbf{Adv}_{\phi[f]}^{\mathrm{wcr}}(\mathcal{A})\end{aligned}$$

$$\mathbf{Adv}_{\phi[f]}^{\mathrm{wcr}}(\mathcal{A}) \leq 2q(\log q+2)\mathbf{Adv}_f^{\mathrm{mac}}(\mathcal{B})$$

Thank You