Cryptanalysis of SFLASH with Slightly Modified Parameters

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It is recommended by the NESSIE European Consortium since 2003

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More precisely...

- SFLASH is some instance of C^{*-} schemes [PGC98]
- All C^{*-} schemes are currently considered secure
- We show that a large class of C^{*-} schemes is insecure
- This class is defined by the non-coprimality of two parameters
- The attack does not apply to the parameters of SFLASH, but the choice of SFLASH parameters was not justified

Organisation of the talk

- A few basics about multivariate schemes
- Description of C^{*-} schemes
- Basic strategy for attacking C^{*-} schemes
- Description of the attack

Multivariate Schemes

- A family of asymmetric schemes
- Hard problems involve MQ polynomials over a finite field $\mathbb{F}_q$
- e.g. solving an MQ system is NP-hard and currently requires exponential time and memory on average

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The Generic Multivariate Construction

- Hiding an easily invertible function using linear transforms

$$\boldsymbol{P} = T \circ P \circ S$$

- Schemes differ from the type of easy function embedded

The C^* Scheme

C^* was proposed by [MI88] and broken by Patarin in 95

Short Description of C^*

- The internal function is a monomial over $\mathbb{F}_{q^n}$

$$P(x) = x^{1+q^\theta} = x.x^{q^\theta}$$

- $\mathbb{F}_{q^n}$ is a n-dimension vector space over $\mathbb{F}_q$, isomorphic to $(\mathbb{F}_q)^n$
- Since a q-powering is linear in $\mathbb{F}_{q^n}$, $P(x)$ is quadratic
- $P(x)$ is an n-tuple of mult. quad. polynomials $(p_1, \ldots, p_n)$

$$p_k(x_1, \ldots, x_n) = \alpha_{12}x_1x_2 + \alpha_{13}x_1x_3 + \ldots$$

- P can be inverted by raising to the inverse power of $1 + q^\theta$
- $\boldsymbol{P} = T \circ P \circ S$ is the public key

The attack by Patarin on C^*

- Any element x and $y = P(x)$ satisfy

$$y^{q^\theta - 1} = x^{(q^\theta+1)(q^\theta-1)} \qquad \Longrightarrow \qquad x.y^{q^\theta} - y.x^{q^{2\theta}} = 0$$

- Consequence : plain and cipher texts are bilinearly related
- These bilinear equations can be determined using pairs (x, y)
- Then, for any specified value y, x is solution of a system of linear equations

C^{*-} Schemes

C^{*-} schemes are C^* schemes with a truncated public key [PGC98]

Construction of a C^{*-} scheme

(n, θ, r) are the parameters of the scheme

1. Generate a C^* with parameters (n, θ) : $P(x) = x^{1+q^\theta}$
2. Remove the last r polynomials from the public key

$$T \circ P \circ S = \begin{cases} \boldsymbol{p}_1(x_1, \ldots, x_n) \\ \vdots \\ \vdots \\ \boldsymbol{p}_n(x_1, \ldots, x_n) \end{cases} \overset{\Pi}{\longmapsto} \begin{cases} \boldsymbol{p}_1(x_1, \ldots, x_n) \\ \vdots \\ \boldsymbol{p}_{n-r}(x_1, .., x_n) \end{cases} = \Pi \circ \boldsymbol{P}$$

Signing with a C^{*-} scheme

1. Append r random bits μ to the message m to be signed
2. Find a preimage σ of (m, μ) by $T \circ P \circ S$ using S, T
3. Such a preimage always exists since a C^* monomial is bijective
4. σ is a valid signature since $\Pi \circ \boldsymbol{P}(\sigma) = m$

Choosing Parameters

Parameters (n, θ) must define a bijective C^*

$$P(x) = x^{1+q^\theta}$$

- P is bijective when $\gcd(q^\theta + 1, q^n - 1) = 1$ (q even)
- This condition is equivalent to n/d odd where $d = \gcd(n, \theta)$

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$q^r \geq 2^{80}$ to avoid a possible recomposing attack from [PGC98]

Proposed Instantiations

The first version of SFLASH was a tweaked C^{*-} scheme

- S, T taken over $\mathbb{F}_2$ rather than $\mathbb{F}_q$ to make the key smaller
- This specificity could be exploited for an attack [GM02]

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Standard Instantiations

	q	n	θ	d	r	Length	PubKey Size
FLASH	2^8	29	11	1	11	296 bits	18 Ko
SFLASHv2 [NESSIE]	2^7	37	11	1	11	259 bits	15 Ko
SFLASHv3	2^7	67	33	1	11	469 bits	112 Ko

Basic Strategy of our Attack

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Important observation

- Consider a C^* public key $\boldsymbol{P} = T \circ P \circ S$

$$\begin{bmatrix} \boldsymbol{p}_1 \\ \vdots \\ \vdots \\ \boldsymbol{p}_n \end{bmatrix} = \begin{bmatrix} t_{11} & \dots & t_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ t_{n1} & \dots & t_{nn} \end{bmatrix} \begin{bmatrix} (P \circ S)_1 \\ \vdots \\ \vdots \\ (P \circ S)_n \end{bmatrix}$$

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- The C^{*-} public key $\Pi \circ \boldsymbol{P}$ consists of the $n - r$ first rows

$$\begin{bmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_{n-r} \end{bmatrix} = \begin{bmatrix} t_{11} & \dots & t_{1n} \\ \vdots & & \vdots \\ t_{n-r,1} & \dots & t_{n-r,n} \end{bmatrix} \begin{bmatrix} (P \circ S)_1 \\ \vdots \\ \vdots \\ (P \circ S)_n \end{bmatrix}$$

Important observation 2

- If we could regenerate r *new* linear combinations

$$\begin{bmatrix} \boldsymbol{p}'_1 \\ \vdots \\ \boldsymbol{p}'_r \end{bmatrix} = \begin{bmatrix} t'_{11} & \dots & t'_{1n} \\ \vdots & & \vdots \\ t'_{r1} & \dots & t'_{rn} \end{bmatrix} \begin{bmatrix} (P \circ S)_1 \\ \vdots \\ (P \circ S)_n \end{bmatrix}$$

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- then adding them to $\Pi \circ \boldsymbol{P}$ might complete a full C^* key :

$$\boldsymbol{P}' = \begin{bmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_{n-r} \\ \boldsymbol{p}'_1 \\ \vdots \\ \boldsymbol{p}'_r \end{bmatrix} = \begin{bmatrix} t_{11} & \dots & t_{1n} \\ \vdots & & \vdots \\ t_{n-r,1} & & t_{n-r,n} \\ t'_{11} & \dots & t'_{1n} \\ \vdots & & \vdots \\ t'_{r1} & \dots & t'_{rn} \end{bmatrix} \begin{bmatrix} (P \circ S)_1 \\ \vdots \\ \vdots \\ (P \circ S)_n \end{bmatrix}$$

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- This C^* public key $\boldsymbol{P}'$ coincides with the original one $\boldsymbol{P}$ on the first $n-r$ coordinates :

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Goal

Find a way to generate new linear combinations of the hidden function $P \circ S$

Basic Strategy 2

A recomposing attack through injection of *commuting* maps !

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We look for pairs of linear maps (M, N) "commuting" with the internal function :

$$P \circ M = N \circ P$$

Then, composing $\Pi \circ \boldsymbol{P}$ with the conjugate of M

$$\boldsymbol{M} = S^{-1} \circ M \circ S$$

generates new linear combinations :

$$\begin{aligned}(\Pi \circ T \circ P \circ S) \circ \boldsymbol{M} &= \Pi \circ T \circ (P \circ M) \circ S \\ &= \Pi \circ T \circ (N \circ P) \circ S \\ &= (\Pi \circ T \circ N) \circ P \circ S\end{aligned}$$

In C^*, P is multiplicative and $M_\xi : x \mapsto \xi.x$ are commuting maps.

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Goal

Find a way to discover some maps $\boldsymbol{M}_\xi$ conjugates of M_ξ :

$$\boldsymbol{M}_\xi = S^{-1} \circ M_\xi \circ S$$

The Differential of C^*

FGS05 : Differential Cryptanalysis for Multivariate Schemes

The differential of a quadratic function P is :

$$DP(a, x) = P(a + x) - P(x) - P(a) + P(0)$$

- DP is bilinear and symmetric in (a, x)
- If $\boldsymbol{P} = T \circ P \circ S$ then $D\boldsymbol{P} = T \circ DP(S, S)$

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The differential of a C^* monomial

$$DP(a,x) = a^{q^\theta} x + a x^{q^\theta} = a^{q^\theta+1}\left(\frac{x}{a}\right) + a^{q^\theta+1}\left(\frac{x}{a}\right)^{q^\theta}$$

Letting $L(\xi) = \xi + \xi^{q^\theta}$, we have :

$$DP(a,x) = P(a).L\left(\frac{x}{a}\right)$$

The Differential of C^*

Notable Consequence

- For any element ξ in $\ker(L)$,

$$DP(a, \xi.a) = P(a).L\left(\frac{\xi.a}{a}\right) = P(a).L(\xi) = 0$$

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$$DP(a, M(a)) = 0$$

- Considering the differential of this equation, these maps satisfy

$$DP(a, M(x)) + DP(M(a), x) = 0$$

M_ξ with ξ in $\ker(L)$ are the *skew-symmetric maps* w.r.t DP.

Skew-symmetric Maps w.r.t the Diff. of the C^* Monomial

The kernel of $L(\xi) = \xi + \xi^{q^\theta}$

- The non-zero elements of $\ker(L)$ satisfy : $\xi^{q^\theta - 1} = 1$
- There are $\gcd(q^\theta - 1, q^n - 1) = q^d - 1$ such elements
- Therefore, *ker*(L) is a linear subspace of dimension d

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Skew-symmetric Maps w.r.t the Diff. of the C^* Monomial

- These maps are **multiplications** M_ξ
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- They form a subspace of dimension $d = \gcd(n, \theta)$.
- This subspace is non-trivial when $d > 1$, since scalar multiples of the identity are useless.

Skew-symmetric Maps w.r.t the Diff. of the C^* Pub.Key

- They are the solutions of the linear equation :

$$D\boldsymbol{P}(\boldsymbol{M}(a), x) + D\boldsymbol{P}(a, \boldsymbol{M}(x)) = 0 \qquad (1)$$

where

$$D\boldsymbol{P} = T \circ DP(S, S)$$

- Therefore, those are :

$$\boldsymbol{M}_\xi = S^{-1} \circ M_\xi \circ S \qquad \text{where} \quad M_\xi(x) = \xi.x \text{ and } \xi \in \ker(L)$$

- Equation (1) : $\simeq n^3$ linear equations in n^2 unknowns over $\mathbb{F}_q$: ($\simeq n^2/2$ lin.indep (a, x) and n coord. of $D\boldsymbol{P}$)

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  We might not need all coordinates of $\boldsymbol{P}$ to recover the $\boldsymbol{M}_\xi$!

- If we are only given the first $n-r$ coordinates of $\boldsymbol{P}$:

$$\Pi \circ D\boldsymbol{P}(\boldsymbol{M}(a), x) + \Pi \circ D\boldsymbol{P}(a, \boldsymbol{M}(x)) = 0$$

gives $(n-r)n(n-1)/2$ linear equations in n^2 unknowns

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- We expect no other solutions when :

$$(n - r)\frac{n(n - 1)}{2} \geq n^2 - d$$

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- We expect no other solutions when :

$$(n - r)\frac{n(n-1)}{2} \geq n^2 - d$$

- Hence, heuristically, the $\boldsymbol{M}_\xi$ are the only solutions up to :

$$r^*_{max} = n - \left\lceil 2\frac{n^2 - d}{n(n-1)} \right\rceil = n - 3$$

- The actual value r_{max} is very close to the heuristical r^*_{max} :

n	36	36	38	39	39	40	42	42	44
θ	8	12	10	13	9	8	12	14	12
d	4	12	2	13	3	8	6	14	4
r^*_{max}	33	33	35	36	36	37	39	39	41
r_{max}	33	32	35	35	36	37	39	38	41

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The skew-symmetric maps can be recovered from as few as 3 or 4 coordinates of the public key !

Recovering a Full C^* Public Key

Using a single non-trivial $\boldsymbol{M}_\xi$, up to $r = n/2$

1. We complete $\Pi \circ \boldsymbol{P}$ using r coordinates of $\Pi \circ \boldsymbol{P} \circ \boldsymbol{M}_\xi$.

$$\left\{ \begin{array}{l} \Pi \circ \boldsymbol{P} \\ (\Pi \circ \boldsymbol{P} \circ \boldsymbol{M}_\xi)_{1 \to r} \end{array} \right. = \left[\begin{array}{l} \Pi \circ T \\ (\Pi \circ T \circ M_{P(\xi)})_{1 \to r} \end{array} \right] \circ P \circ S$$

2. We can check that this is a full C^* public key since Patarin's attack works again.

n	36	36	38	39	39	40	42	42	44
θ	8	12	10	13	9	8	12	14	12
d	4	12	2	13	3	8	6	14	4
r	11	11	11	12	12	12	13	13	13
$C^{*-} \mapsto C^*$	$57s$	$57s$	$94s$	$105s$	$90s$	$105s$	$141s$	$155s$	$155s$

Note : parameters are close to those of SFLASHv2, with the same $q = 2^7$.

Recovering a Full C^* Public Key

Using a whole basis of $\boldsymbol{M}_\xi$

Since we have $d(n-r)$ coordinates available, the overall bound is :

$$r \leq n\left(1 - \frac{1}{d}\right)$$

n	36	36	38	39	39	40	42	42	44
θ	8	12	10	13	9	8	12	14	12
d	4	12	2	13	3	8	6	14	4
r	27	32	19	35	26	35	35	38	33
$C^{*-} \mapsto C^*$	$65s$	$51s$	$112s$	$79s$	$107s$	$95s$	$134s$	$117s$	$202s$

Conclusion

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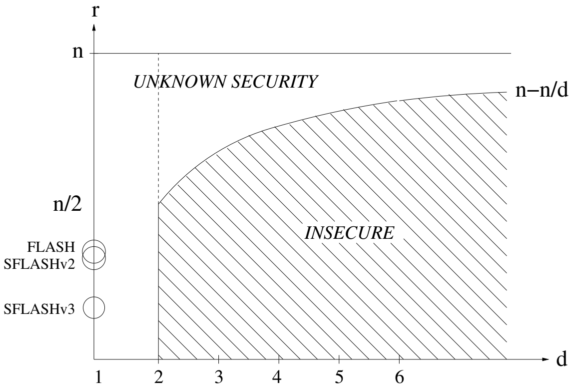

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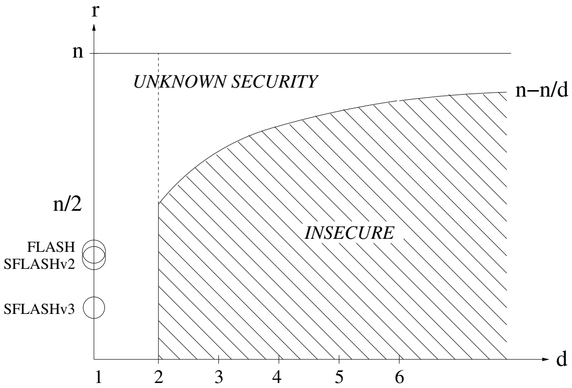
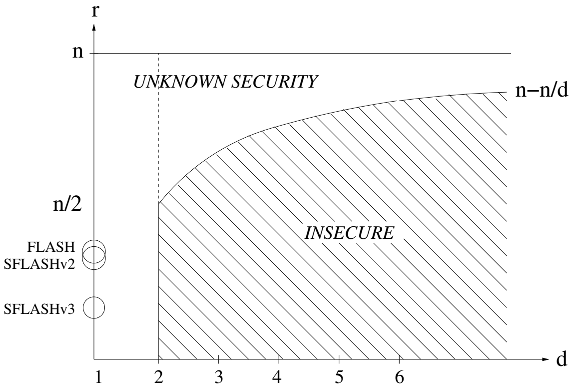


Conclusion

- C^{*-} schemes with $d > 1$ are insecure up to $r = n(1 - \frac{1}{d})$
- The attack does not apply to the case $d = 1$ (but a different way to find multiplications breaks these cases : see Crypto07, joint work with Adi Shamir)



Thank you for your attention !

Questions ?