Breaking and Repairing GCM Security Proofs

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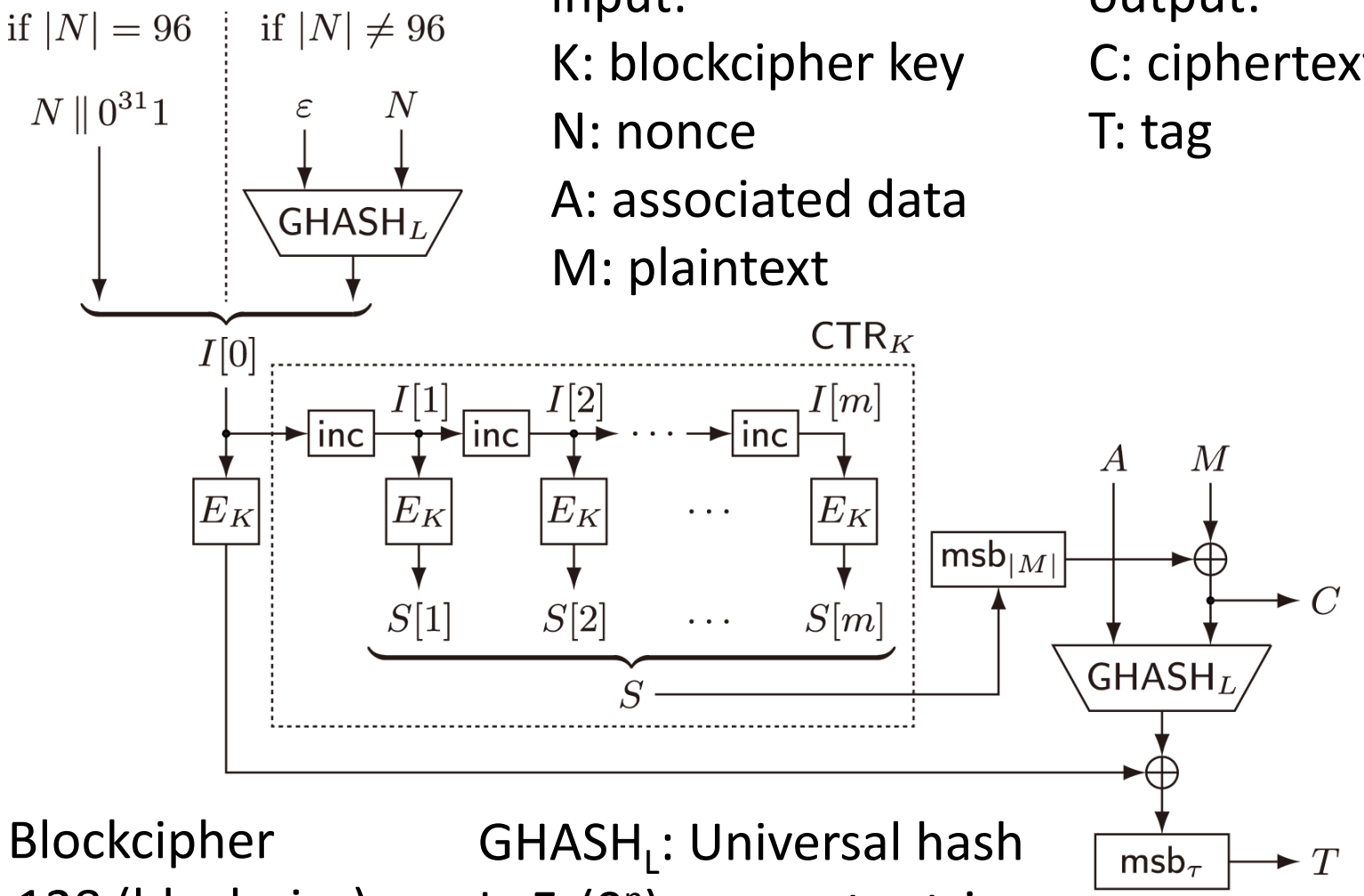
CRYPTO 2012
August 20, 2012, Santa Barbara, USA

GCM

- Galois/Counter Mode
- authenticated encryption mode of 128-bit blockciphers
- designed by McGrew and Viega in 2004 [MV04]
- selected as the NIST recommended authenticated encryption mode in 2007
- widely used in practice
  - ISO/IEC 19772, IEEE P1619.1, NSA Suite B, IETF IPsec, SSH, SSL,…

[MV04] David A. McGrew and John Viega: The Security and Performance of the Galois/Counter Mode (GCM) of Operation. Cryptology ePrint Archive: Report 2004/193 (full version of INDOCRYPT 2004)

Encryption Algorithm of GCM



input:
K: blockcipher key
N: nonce
A: associated data
M: plaintext

output:
C: ciphertext
T: tag

E_K: Blockcipher
n = 128 (block size)

GHASH_L: Universal hash
$L=E_K(0^n)$, ε: empty string

Provable Security Results

- The designers proved the security of GCM [MV04]
- analyzed privacy and authenticity against chosen ciphertext attacks
- Privacy bound:
  - $$\mathbf{Adv}^{\mathrm{priv}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \frac{0.5(\sigma/n+2q)^2}{2^n} + \frac{2q(\sigma/n+2q)\lceil \ell_N/n+1\rceil}{2^n} + \frac{q\lceil \ell/n+1\rceil}{2^\tau}$$
  - Ciphertexts of GCM are indistinguishable from random strings

Provable Security Results

- Authenticity bound:
  - $\mathbf{Adv}^{\mathrm{auth}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \frac{0.5(\sigma/n+2q)^2}{2^n} + \frac{2q(\sigma/n+2q+1)\lceil \ell_N/n+1\rceil}{2^n} + \frac{q\lceil \ell/n+1\rceil}{2^\tau}$
  - GCM is unforgeable

Previous Security Analyses

- [Ferguson ’05]
  - forgery attacks when the tag is short
- [Joux ’06]
  - key recovery attacks on GCM (nonce reuse), forgery attacks on the draft NIST version of GCM
- [Handschuh, Preneel ’08]
  - a weak key, forgery attacks
- [Saarinen ’12]
  - many weak keys, forgery attacks

Previous Security Analyses

- It is widely considered that the provable security results of GCM are sound
  - in the sense that these attacks do not contradict the claimed security bounds, and that no flaw in the proofs has been identified
  - show the tightness of the security bounds
  - outside the security model (e.g., nonce reuse)

Equation Over GF(2^{128})

- defined by the irreducible polynomial $p(x) = 1+x+x^2+x^7+x^{128}$ (used in GCM)
- the multiplicative identity element is 0x80…0 (10…0 in binary)

$$U \cdot L^2 \oplus V \cdot L \oplus 0x0\ldots01 = U' \cdot L^2 \oplus V \cdot L$$

U = 0x00000000 00000000 02000000 00000000 (128 bits)
U'= 0x00000000 00000000 06000000 00000000 (128 bits)
V = 0x00000000 00000000 00000000 00000048 (128 bits)

- How many solutions (in L) do we have?

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- How many solutions (in L) do we have?
  - at most 2 solutions (actually one solution)

Increment Function in GCM

- inc(X || Y) = X || (Y+1 mod 2^{32})
  - |X| = 96, |Y|=32
  - inc(0x0…01) = 0x0…02

$$U \cdot L^2 \oplus V \cdot L \oplus 0x0\ldots01 = U' \cdot L^2 \oplus V \cdot L$$

$$inc(U \cdot L^2 \oplus V \cdot L) = U' \cdot L^2 \oplus V \cdot L$$

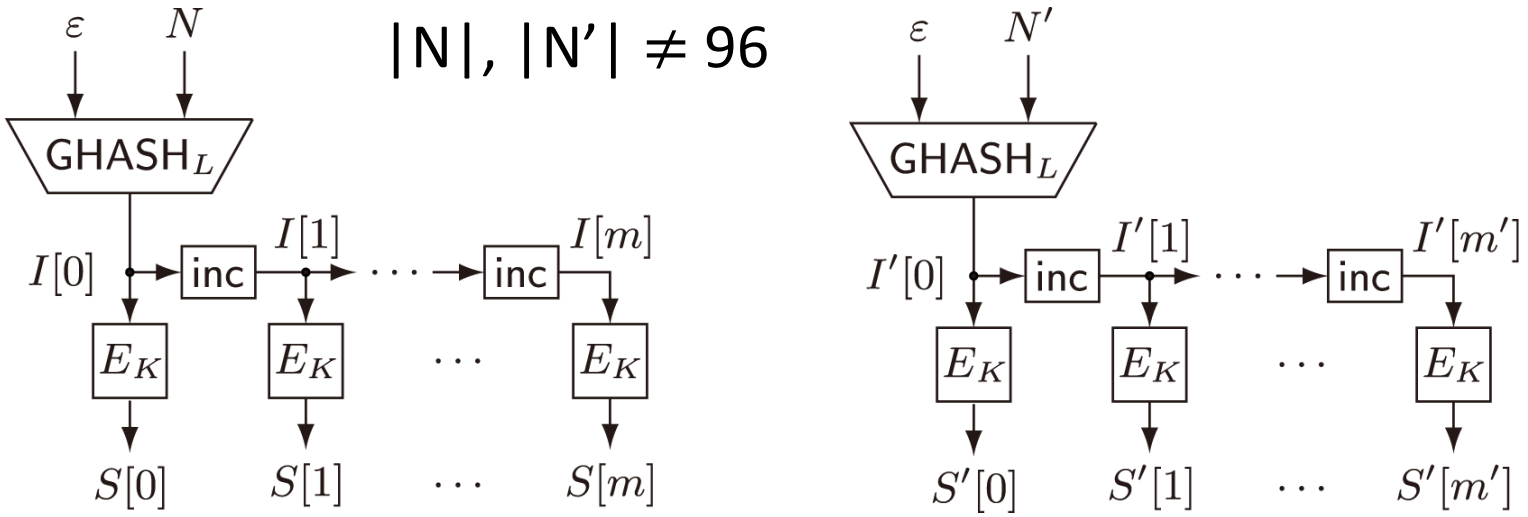
- How many solutions (in L) do we have?
  - Note: LHS may not be a degree 2 polynomial over GF(2^{128})

List of Solutions

0x7f6db6d2db6db6db6db6db6492492492, 0x7f6db6dadb6db6db6db6db6492492492,
0x81b6db776db6db6db6db6dadb6db6db6, 0x81b6db676db6db6db6db6dadb6db6db6,
0xbe00003c00000000000003ffffffff, 0xbe00001c00000000000003ffffffff,
0xc16db6aadb6db6db6db6db1b6db6db6d, 0xc16db6eadb6db6db6db6db1b6db6db6d,
0x3fb6db876db6db6db6db6d5249249249, 0x3fb6db076db6db6db6db6d5249249249,
0x000001dc0000000000001c000000000, 0x000000dc0000000000001c000000000,
0x7f6db56adb6db6db6db6d8e492492492, 0x7f6db76adb6db6db6db6d8e492492492,
0x81b6dc076db6db6db6db6aadb6db6db6, 0x81b6d8076db6db6db6db6aadb6db6db6,
0xbe000edc000000000000e3ffffffff, 0xbe0006dc000000000000e3ffffffff,
0xc16dab6adb6db6db6db6c71b6db6db6d, 0xc16dbb6adb6db6db6db6c71b6db6db6d,
0x3fb6e0076db6db6db6db555249249249, 0x3fb6c0076db6db6db6db555249249249,
0x000076dc0000000000071c000000000, 0x000036dc0000000000071c000000000,
0x7f6d5b6adb6db6db6db638e492492492, 0x7f6ddb6adb6db6db6db638e492492492,
0x81b700076db6db6db6daaaadb6db6db6, 0x81b600076db6db6db6daaaadb6db6db6,
0xbe03b6dc000000000038e3ffffffff, 0xbe01b6dc000000000038e3ffffffff,
0xc16adb6adb6db6db6db1c71b6db6db6d, 0x00000040000000000000000000000000

- Answer: 32 solutions

Counter Collision



- A counter collision is a bad event: I[1] = I'[1], I[2] = I'[1], …
  - xor of two ciphertexts = xor of two plaintexts
  - the information about plaintexts is leaked
- We need to show that $Pr_L[\ Coll_L(r, N, N')\ ]$ is small
  - $Coll_L(r, N, N')$: $inc^r(\ GHASH_L(\varepsilon, N)\ ) = GHASH_L(\varepsilon, N')$

$GHASH_L(\varepsilon, N)$

- universal hash function
- $N \,||\, 0\ldots0 \,||\, |N|_n = (X[1],\ldots,X[x])$
- $GHASH_L(\varepsilon, N) = X[1] \cdot L^x \oplus X[2] \cdot L^{x-1} \oplus \ldots \oplus X[x] \cdot L$
- N = 0x00000000 00000000 02 (72 bits)
- $GHASH_L(\varepsilon, N)$

  $= 0x00000000\ 00000000\ 02000000\ 00000000 \cdot L^2$

  $\oplus\ 0x00000000\ 00000000\ 00000000\ 00000048 \cdot L$

  $= U \cdot L^2 \oplus V \cdot L$
- N’ = 0x00000000 00000000 06 (72 bits)
- $GHASH_L(\varepsilon, N') = U' \cdot L^2 \oplus V \cdot L$

$Pr_L[\ Coll_L(r, N, N')\ ]$ Is Small

- [Lemma 3, MV04]

  $Pr_L[\ Coll_L(r, N, N')\ ] \leq \max\{ d, d' \} / 2^n$

  where $d = \deg(\ GHASH_L(\varepsilon, N)\ )$, $d' = \deg(\ GHASH_L(\varepsilon, N')\ )$

- The lemma claims

  "$inc(\ U \cdot L^2 \oplus V \cdot L\ ) = U' \cdot L^2 \oplus V \cdot L$ has at most 2 solutions."

$\Pr_L[\ \mathrm{Coll}_L(r, N, N')\ ]$ Is Small

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- The lemma claims

  "$\mathrm{inc}(\ U \cdot L^2 \oplus V \cdot L\ ) = U' \cdot L^2 \oplus V \cdot L$ has at most 2 solutions."

- [Lemma 3, MV04] is incorrect
  - used in both the privacy proof and the authenticity proof
  - both proofs contain a flaw

More Observation

$$\mathrm{inc}(\, U \cdot L^2 \oplus V \cdot L \,) = U' \cdot L^2 \oplus V \cdot L \qquad \text{(A)}$$
$$\mathrm{inc}^2(\, U \cdot L^2 \oplus V \cdot L \,) = U' \cdot L^2 \oplus V \cdot L \qquad \text{(B)}$$
$$\mathrm{inc}^4(\, U \cdot L^2 \oplus V \cdot L \,) = U' \cdot L^2 \oplus V \cdot L \qquad \text{(C)}$$
$$\mathrm{inc}^0(\, U \cdot L^2 \oplus V \cdot L \,) = U' \cdot L^2 \oplus V \cdot L \qquad \text{(D)}$$

- Number of solutions
  - (A): 32, (B): 31, (C): 30, (D): 1
- 94 solutions are all distinct
- $\Pr_L[$ (A) or (B) or (C) or (D) $] \geq 94/2^{128}$

Distinguishing Attack

- The observation can be translated into a distinguishing attack on GCM[Rand(n),τ]: GCM with a random function R (instead of E_K)
  - by simply observing if the event occurs in ciphertexts
  - $\mathrm{Adv}^{\mathrm{priv}}_{\mathrm{GCM[Rand(n),}\tau\mathrm{]}}(A) \geq 94/2^{128}$
- The attack does not contradict the overall privacy bound, but it invalidates a part of it
  - $$\mathbf{Adv}^{\mathrm{priv}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \frac{0.5(\sigma/n+2q)^2}{2^n} + \underline{\frac{2q(\sigma/n+2q)\lceil \ell_N/n+1\rceil}{2^n}} + \frac{q\lceil \ell/n+1\rceil}{2^\tau}$$
  - The second term: $\mathrm{Adv}^{\mathrm{priv}}_{\mathrm{GCM[Rand(n),}\tau\mathrm{]}}(A) \leq 80/2^{128}$

Remarks

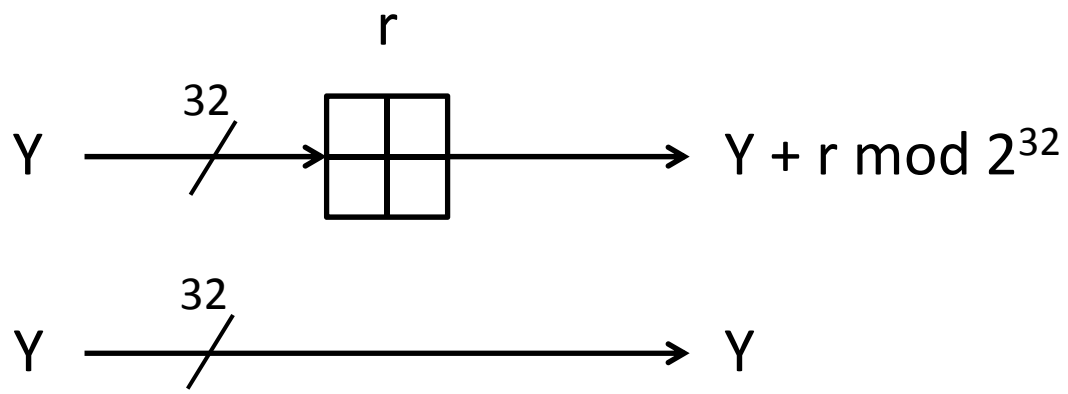
- The attack does not break GCM
  - Our attack does not contradict the overall privacy bound
  - it invalidates only a part of it
  - the attack also invalidates a part of the authenticity bound
- The success probability of the attack is small
  - The practical implication is limited
- The attack does not work if the nonce length is restricted to 96 bits (required or recommended by many standards mainly for efficiency reasons)

Can We Repair the Proofs?

- without modifying the original specification
- $\Pr_L[\ \mathrm{Coll}_L(r, N, N')\ ] \leq ?$
  - introduce a combinatorial problem
  - relation to the proof
  - approaches to solve the problem
  - new privacy and authenticity bounds

Combinatorial Problem

$\mathbf{Y}_r = \{ (Y + r \bmod 2^{32}) \oplus Y \mid Y \text{ is in } \{0,1\}^{32} \}$
$\alpha_r = \# \mathbf{Y}_r$
problem: determine $\alpha_{max} = \max\{ \alpha_r \mid 0 \le r \le 2^{32}-1 \}$



α_r = the number of possible non-zero xor differences of $Y + r \bmod 2^{32}$ and Y when Y ranges over $\{0,1\}^{32}$

Relation to the Proof

- $\mathrm{Coll}_L(r, N, N')$: $\mathrm{inc}^r(\ \mathrm{GHASH}_L(\varepsilon, N)\ ) = \mathrm{GHASH}_L(\varepsilon, N')$

- if we can replace $\mathrm{inc}^r(\ \mathrm{GHASH}_L(\varepsilon, N)\ )$ by $\mathrm{GHASH}_L(\varepsilon, N) \oplus C$, then we can derive the upper bound on $\Pr_L[\ \mathrm{Coll}_L(r, N, N')\ ]$

$$\mathrm{GHASH}_L(\varepsilon, N) \oplus C = \mathrm{GHASH}_L(\varepsilon, N') \qquad (*)$$

- but C depends on r and $\mathrm{GHASH}_L(\varepsilon, N)$
- $\alpha_r = \#\{\ (Y + r \bmod 2^{32}) \oplus Y \mid Y \text{ is in } \{0,1\}^{32}\ \}$ represents the number of possibilities of C
- For each C, we know the number of solutions of (*)

Relation to the Proof

- Towards a new version of [Lemma 3, MV04]
- Lemma

  For each $0 \le r \le 2^{32}-1$

  $$\Pr_L[\ \mathrm{Coll}_L(r, N, N')\ ] \le \alpha_r \max\{\ d, d'\ \} / 2^n$$

  where $d = \deg(\ \mathrm{GHASH}_L(\varepsilon, N)\ )$, $d' = \deg(\ \mathrm{GHASH}_L(\varepsilon, N')\ )$

- For any $0 \le r \le 2^{32}-1$

  $$\Pr_L[\ \mathrm{Coll}_L(r, N, N')\ ] \le \alpha_{max} \max\{\ d, d'\ \} / 2^n$$

Approaches to Solve the Problem

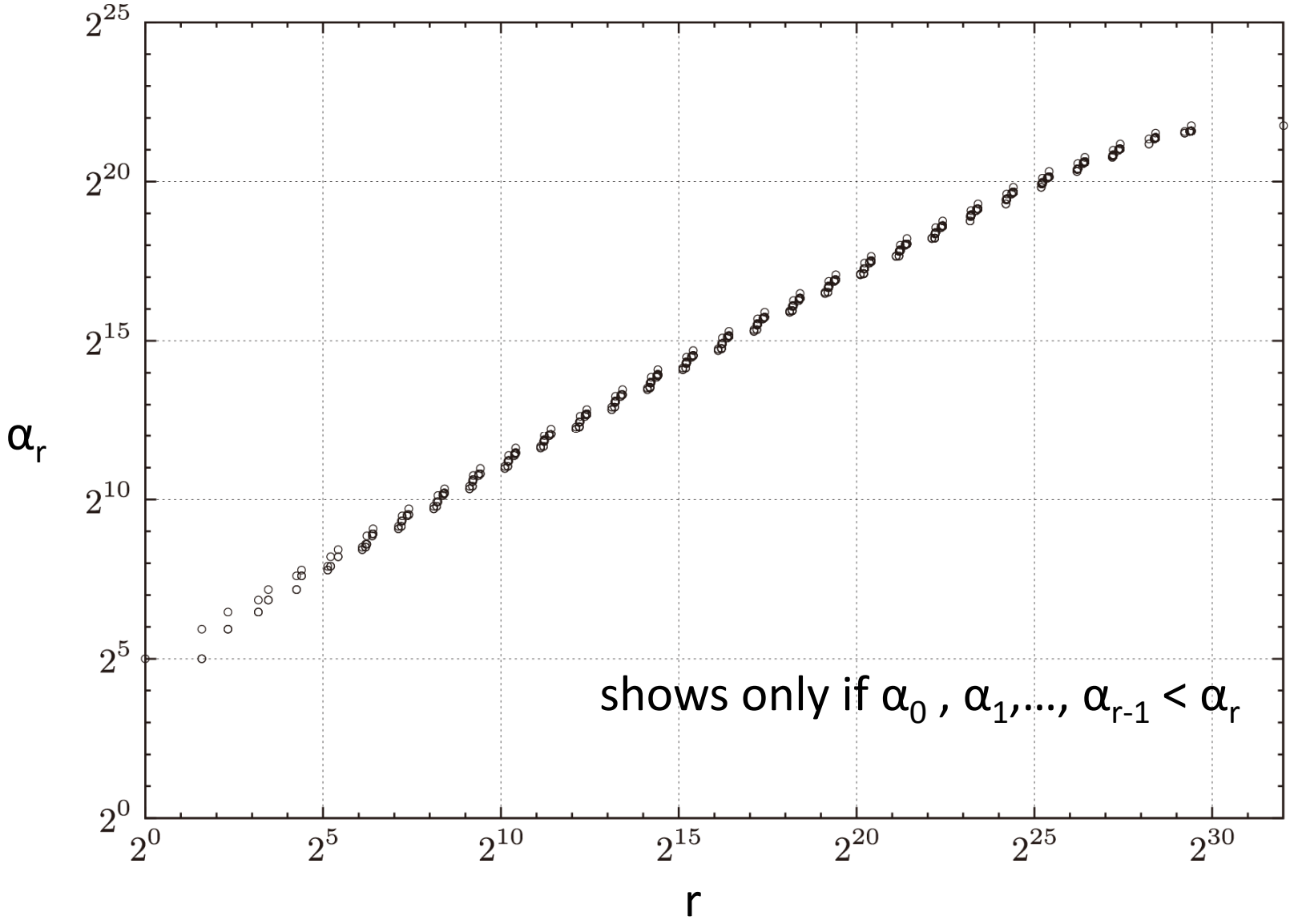
- Make use of tools for the analysis of S-functions [Mouha, et al. ’11, Leurent, ’12]
- Our solution:
  - a recursive formula to compute α_r
  - Proposition

    if $s_\ell = 0$, then $A_\ell = t_\ell A_{\ell-1} + B_{\ell-1}$

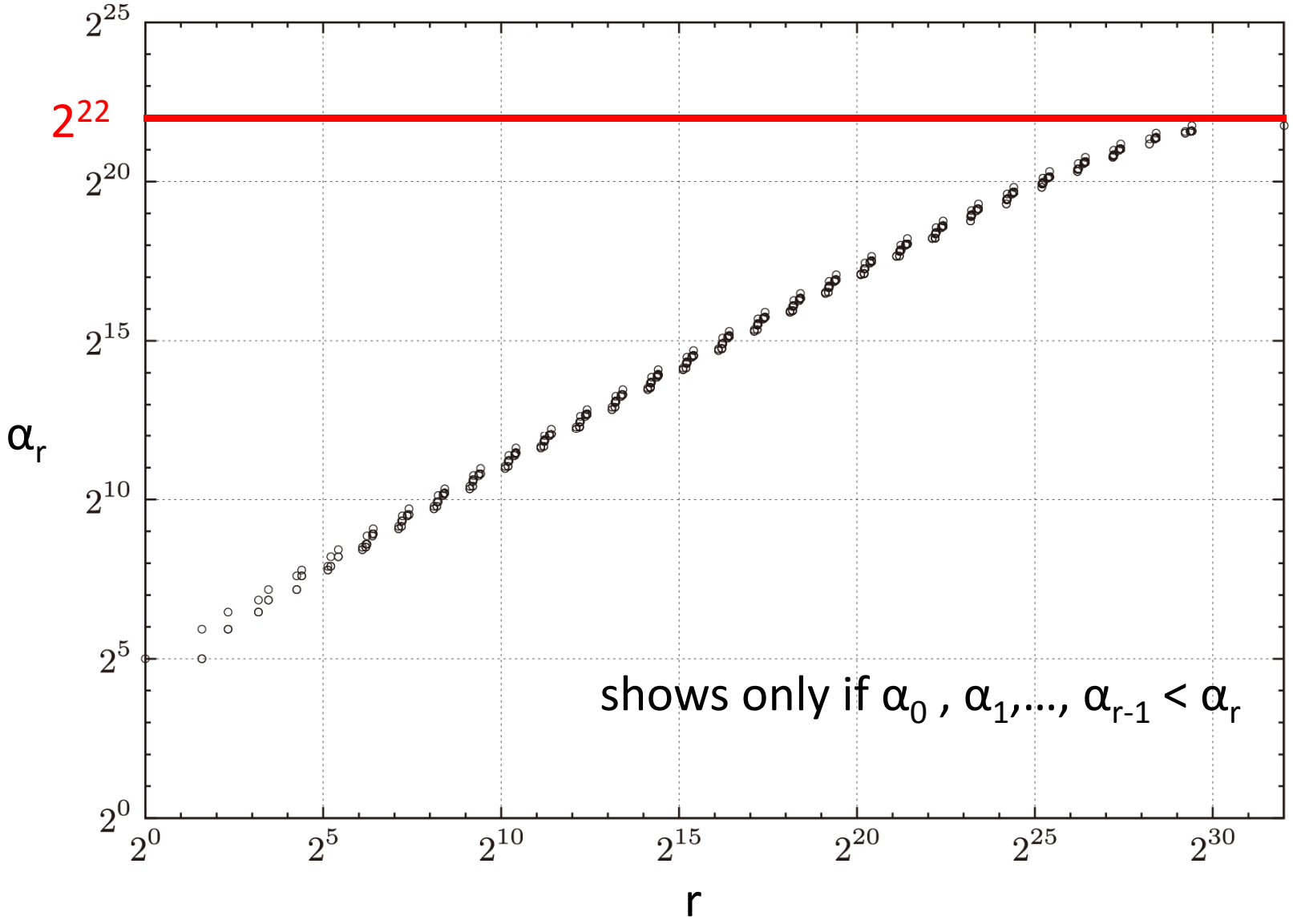
    if $s_\ell \geq 1$, then $A_\ell = s_\ell B_\ell + A_{\ell-1}$

    where $B_j = t_j A_{j-1} + B_{j-1}$ for $0 < j \leq \ell$, $A_j = s_j B_j + A_{j-1}$ for $0 < j \leq \ell-1$, $A_0 = 1$, and $B_0 = 0$
  - can be used to efficiently compute α_r

Graph of α_r



Graph of α_r



$\alpha_{max} = 3524578 \leq 2^{22}$

- $\alpha_{max} = 3524578$ is achieved when

  r = 0x2aaaaaab, 0x55555555, 0xaaaaaaab, 0xd5555555

- New version of [Lemma 3, MV04]

For any $0 \leq r \leq 2^{32}-1$,

$$Pr_L[\ Coll_L(r, N, N')\ ] \leq 2^{22} \max\{ d, d' \} / 2^n$$

where $d = \deg(\ GHASH_L(\varepsilon, N)\ )$, $d' = \deg(\ GHASH_L(\varepsilon, N')\ )$

New Privacy Theorem

- $$\mathbf{Adv}^{\mathrm{priv}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \frac{0.5(\sigma+q+1)^2}{2^n} + \frac{2^{22}q(\sigma+q)(\ell_N+1)}{2^n}$$

- essentially the same as the original privacy bound
  - chosen plaintext attacks
  - main difference is 2^{22}

- If the nonce length is restricted to 96 bits, then

$$\mathbf{Adv}^{\mathrm{priv}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \frac{0.5(\sigma+q+1)^2}{2^n}$$

New Authenticity Theorem

- $$\mathbf{Adv}^{\mathrm{auth}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \frac{0.5(\sigma + q + q' + 1)^2}{2^n} + \frac{2^{22}(q + q' + 1)(\sigma + q)(\ell_N + 1)}{2^n} + \frac{q'(\ell_A + 1)}{2^\tau}$$
- essentially the same as the original authenticity bound
  - main difference is 2^{22}
- If the nonce length is restricted to 96 bits, then

$$\mathbf{Adv}^{\mathrm{auth}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \frac{0.5(\sigma + q + q' + 1)^2}{2^n} + \frac{q'(\ell_A + 1)}{2^\tau}$$

Conclusions

- [Lemma 3, MV04] is not correct
  - the probability of a counter collision is higher than claimed
- The proofs can be repaired
  - new version of [Lemma 3, MV04]
  - new privacy theorem and new authenticity theorem
  - the bounds are worse than the original bounds, but GCM maintains the provable security (both in privacy and authenticity)
  - better bounds if the nonce length is restricted to 96 bits
- Open problem: Can we improve our security bounds?