Analyzing Blockwise Lattice Algorithms using Dynamical Systems

Guillaume Hanrot, Xavier Pujol, Damien Stehlé

ENS Lyon, LIP (CNRS – ENSL – INRIA – UCBL - ULyon)

Context

- Lattices provide exponentially hard problems suitable for public key cryptography.
- Best known attacks on lattice-based cryptosystems rely on blockwise lattice reduction algorithms.
- Understanding these algorithms helps assessing the security of LBC.
- The most widely used reduction algorithm is BKZ.
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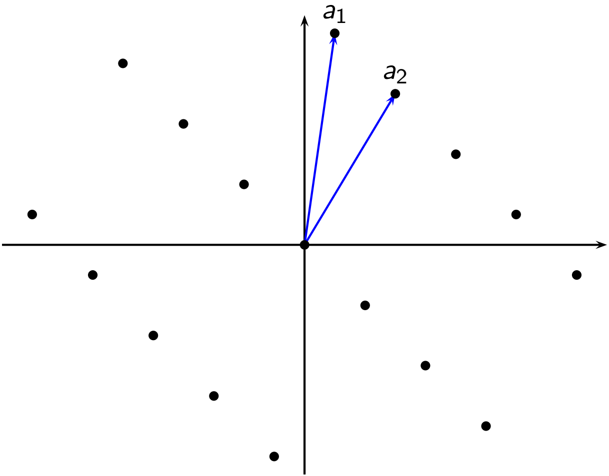

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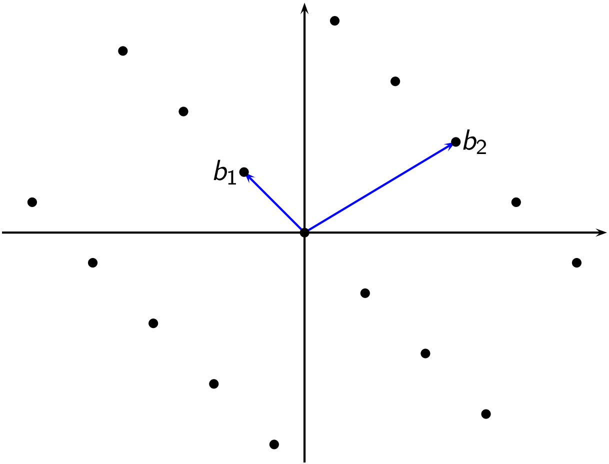

Lattice reduction

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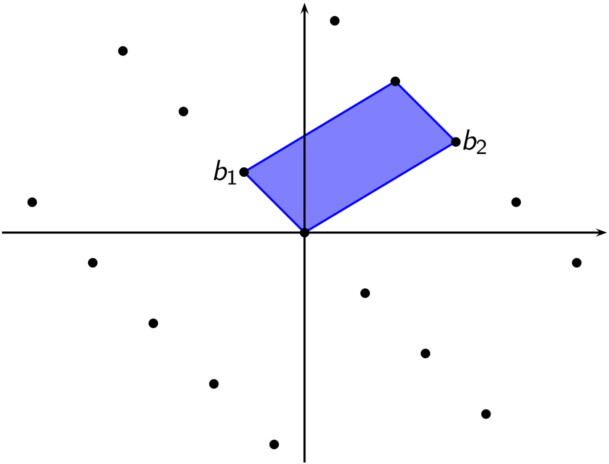

Determinant

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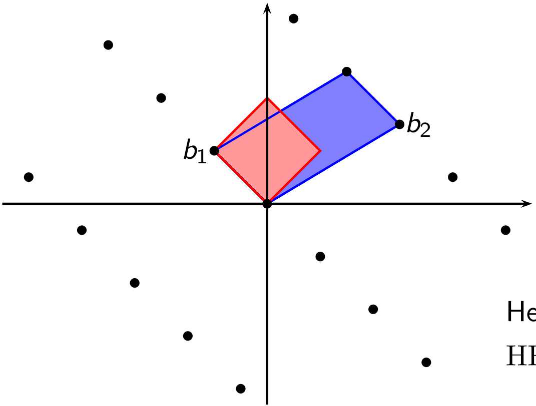

Hermite factor of B:
$$\mathrm{HF}(b_1, \ldots, b_n) = \frac{\|b_1\|}{(\det L)^{1/n}}$$

- Goal of lattice reduction: find a basis with small HF.
- If b_1 is a shortest vector $\neq 0$, then $\mathrm{HF}(b_1, \ldots, b_n) \leq \sqrt{\gamma_n}$, with γ_n = Hermite constant $\leq n$.

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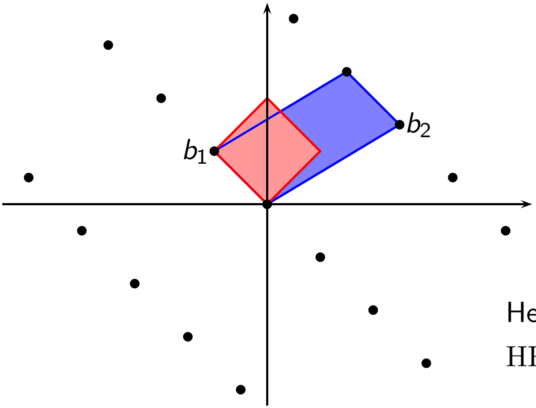

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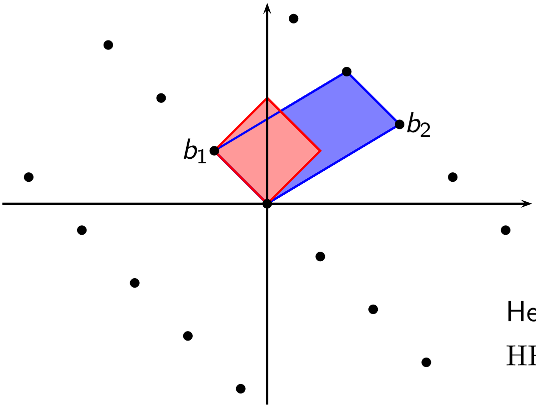

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Hierarchy of lattice reductions in dimension n

$x_i = \log \|b_i^*\|$ for $i \leq n$ ($b_1^*, \ldots, b_n^* =$ Gram-Schmidt basis of B).

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HF: $\sqrt{\gamma_n}$

Time: $2^{O(n)}$

BKZ$_\beta$
Block Korkine-Zolotareff

$\simeq (\gamma_\beta)^{\frac{n}{2\beta}}$

$2^{O(\beta)} \times ?$

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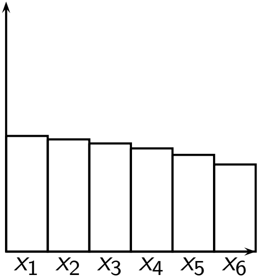

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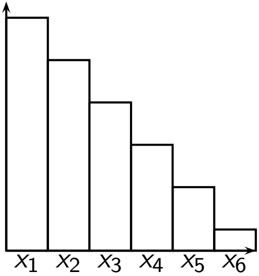

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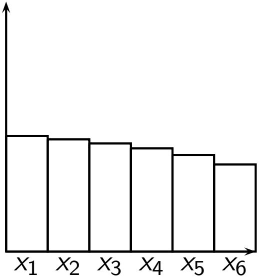

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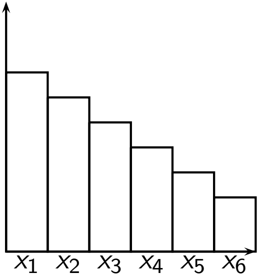

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- Schnorr (1987): first hierarchies between LLL and HKZ.
- Schnorr and Euchner (1994): algorithm for BKZ-reduction.
- Gama and Nguyen (2008): BKZ behaves badly when the block size is ≥ 25.

Other reductions in time $2^{O(\beta)} \times \mathrm{Poly}(n)$:

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Input: B of dimension n.
Repeat ... times
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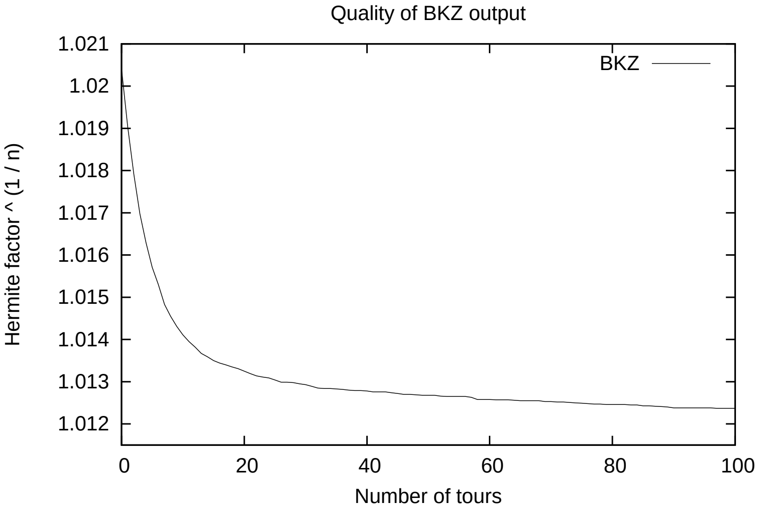
Progress made during the execution of BKZ



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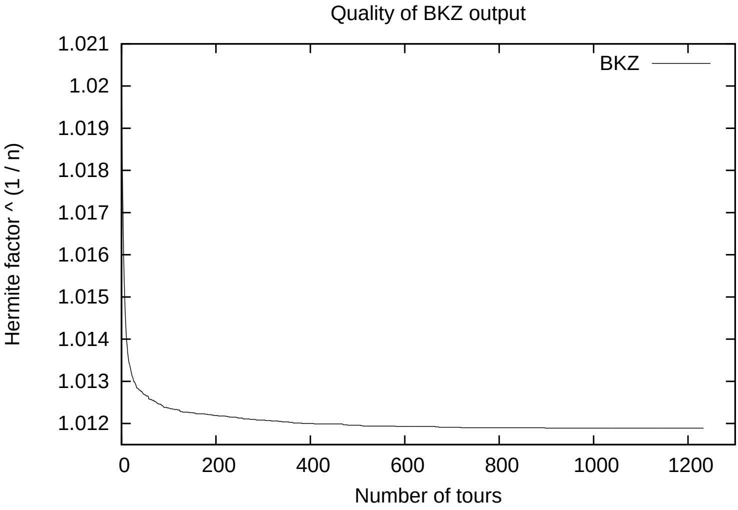

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Our result

γ_β = Hermite constant $\leq \beta$.
L a lattice with basis $(b_1, \ldots, b_n)$.

Theorem

After $\mathcal{O}\left(\frac{n^3}{\beta^2}\left(\log\frac{n}{\epsilon} + \log\log\max\frac{\|b_i\|}{(\det L)^{1/n}}\right)\right)$ *calls to* HKZ_β*, BKZ_β returns a basis C of L such that:*

$$\mathrm{HF}(C) \leq (1+\epsilon)\gamma_\beta^{\frac{n-1}{2(\beta-1)}+\frac{3}{2}}.$$

Sandpile model

- We consider only $x_i = \log \|b_i^*\|$ for $i \leq n$.
- We assume that HKZ-reductions correspond to a fixed pattern.
- The information on the initial x_i's fully determines the x_i's after a call to HKZ.

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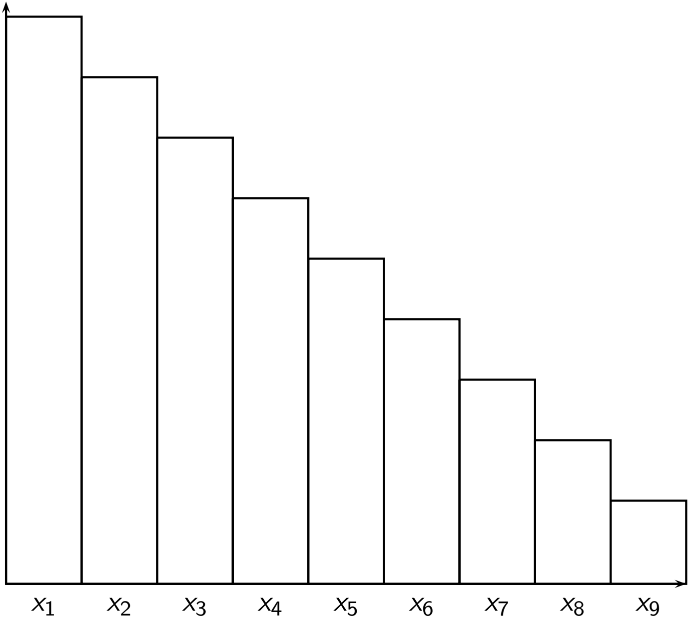
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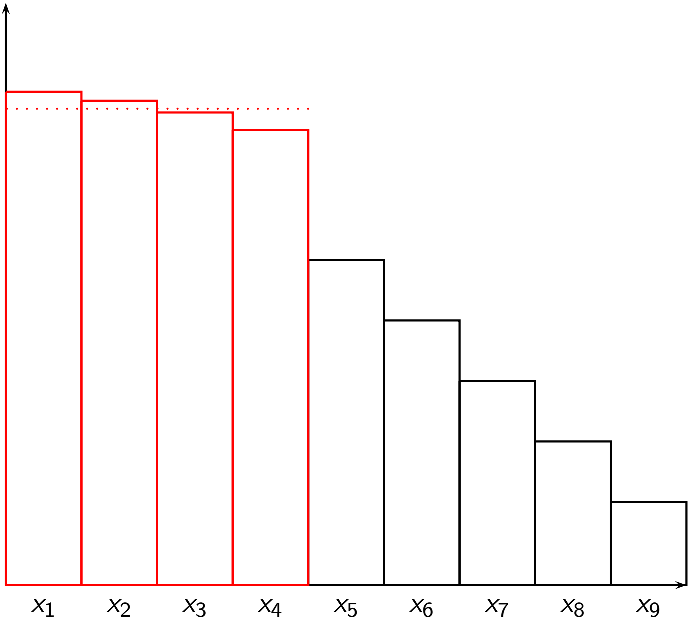
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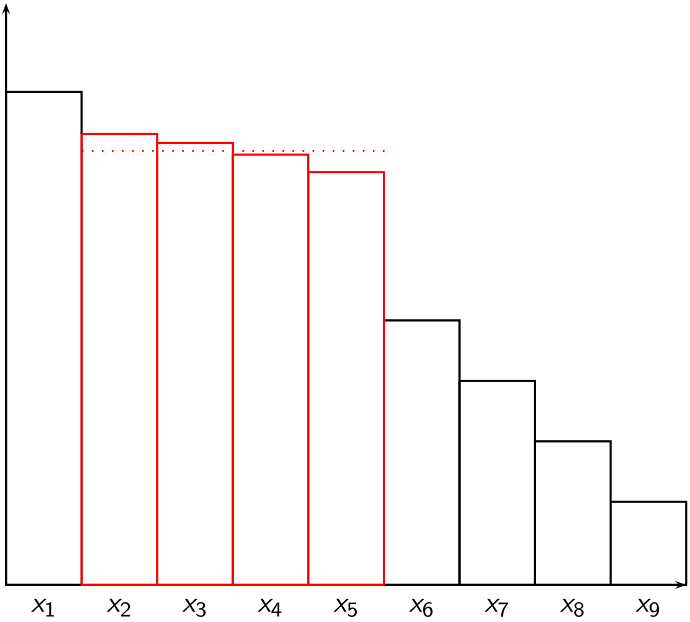
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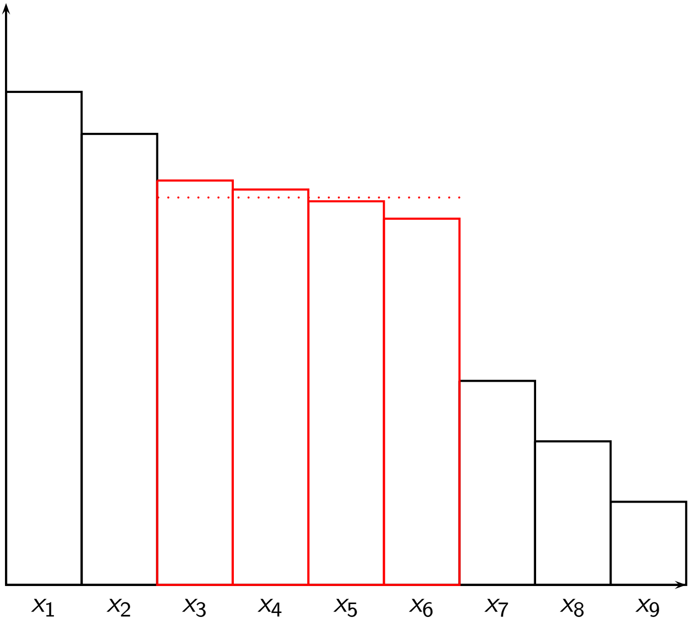
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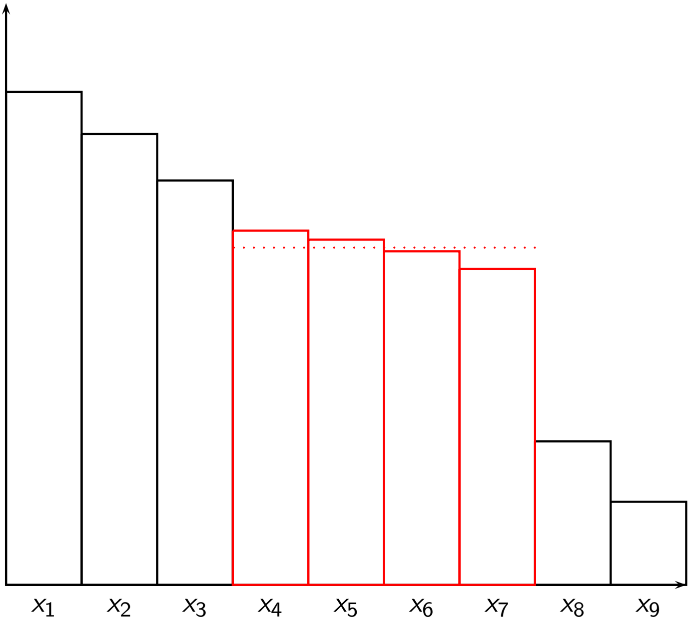
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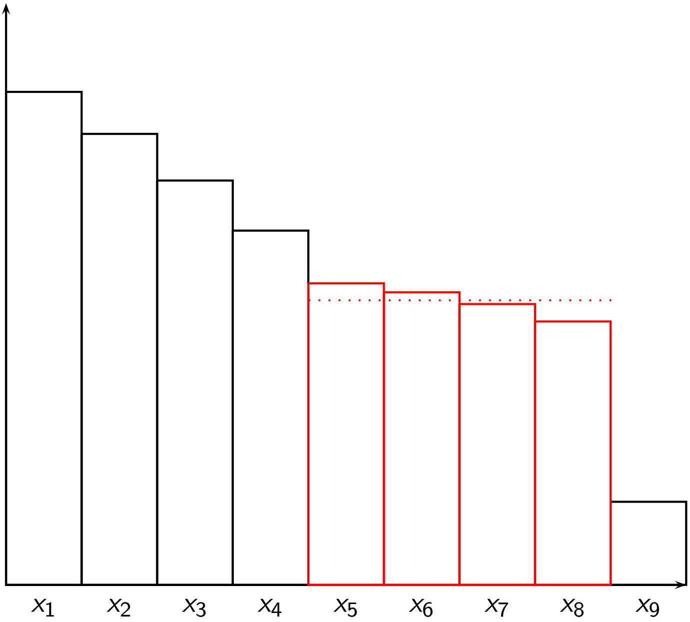
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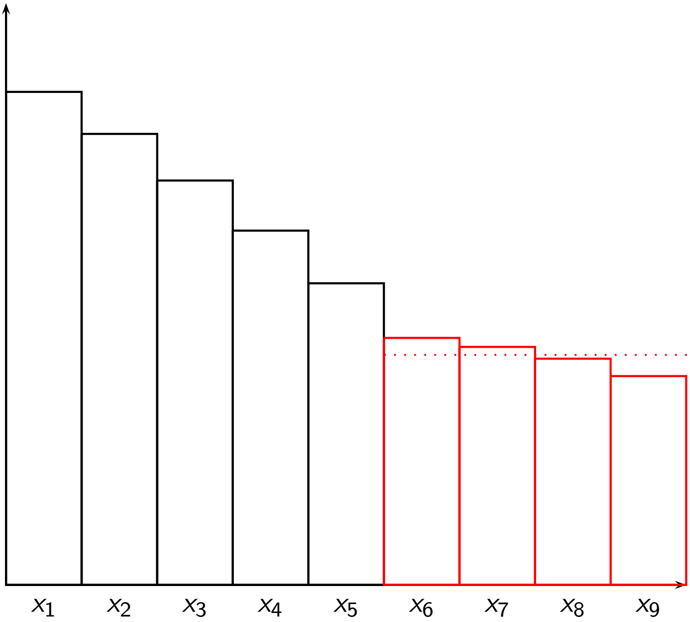
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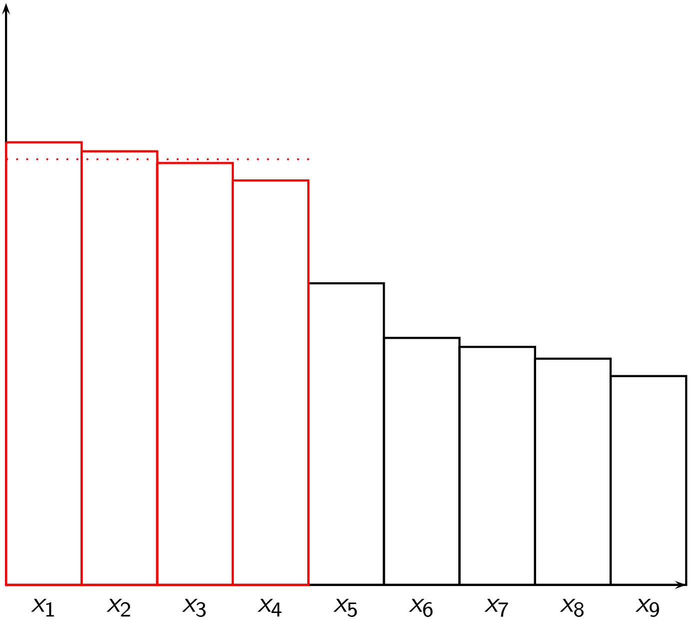

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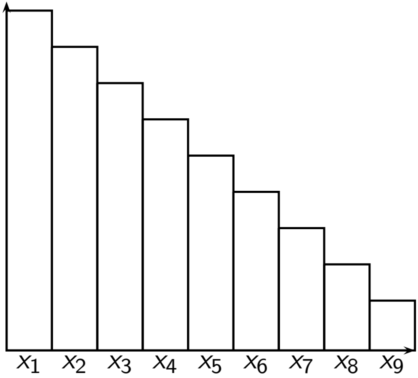
Matrix interpretation



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with $k = n - \beta + 1$

A full tour:

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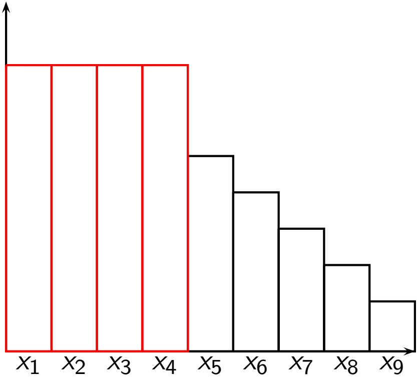

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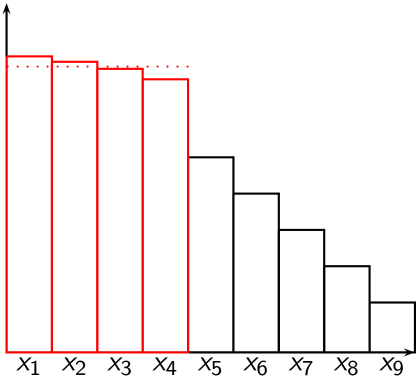

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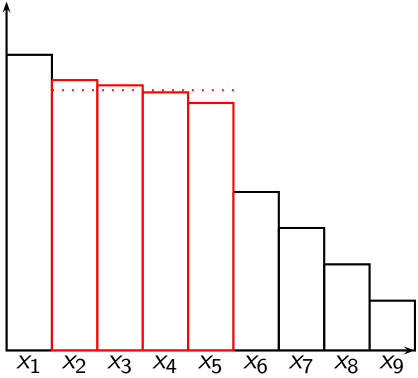

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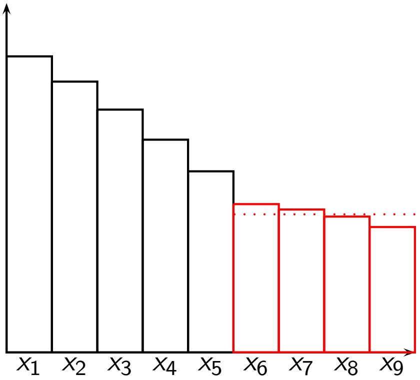

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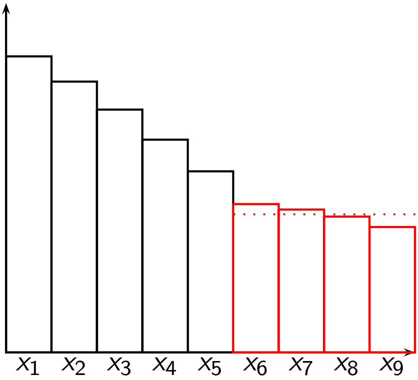

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Quality of the output

Method: study the fixed point of:

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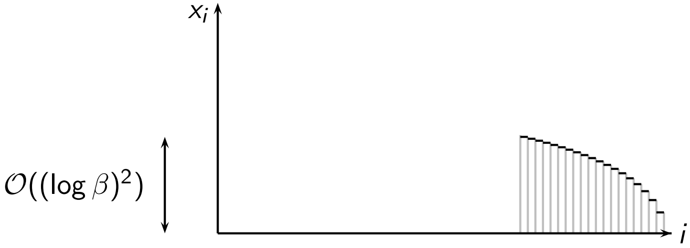
- Asymptotically, line of slope $-\frac{\log \gamma_\beta}{\beta - 1}$.



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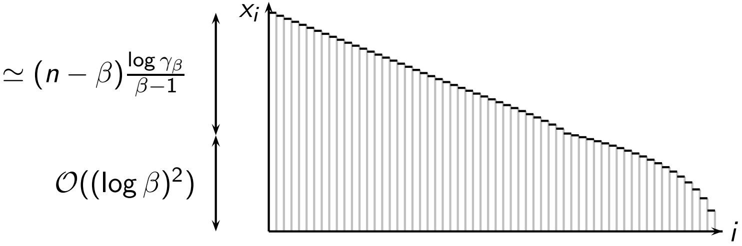

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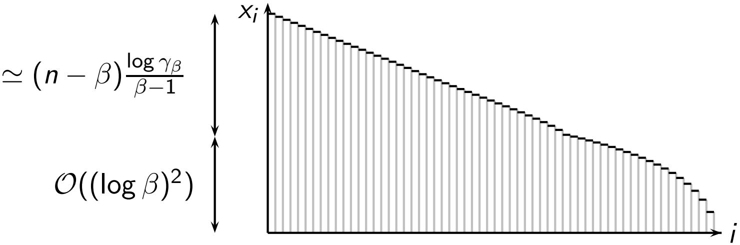

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Result: the largest eigenvalue of $A^T A$ smaller than 1 is

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