Constructing Lossy Trapdoor Functions

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Introduction

Lossy Encryption and Lossy Trapdoor Functions

LTFs from Lossy Encryption

Randomness Dependent Message (RDM) Security

Conclusion and Open Problems

Derandomizing encryption schemes

Trying to build injective trapdoor functions

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Trying to build injective trapdoor functions

CPA secure encryption of x

$\mathsf{Enc}_{pk}(x, r)$

Derandomizing encryption schemes

Trying to build injective trapdoor functions

$\mathrm{Enc}_{pk}(x, f(x))$

Randomness is a function of message

Derandomizing encryption schemes

Trying to build injective trapdoor functions

Is this one-way?

$\text{Enc}_{pk}(x, f(x))$

Randomness is a function of message

Derandomizing encryption schemes

Trying to build injective trapdoor functions

$\mathsf{Enc}_{pk}(x, f(x))$

$\boxed{\text{In general this is a bad idea!}}$

Randomness is a function of message

Encrypting randomness dependent messages is a bad idea

A simple example: using message as randomness

Suppose:

- $\text{Enc}_{pk}(\cdot, \cdot)$ is CPA secure
- Messages and randomness are the same length

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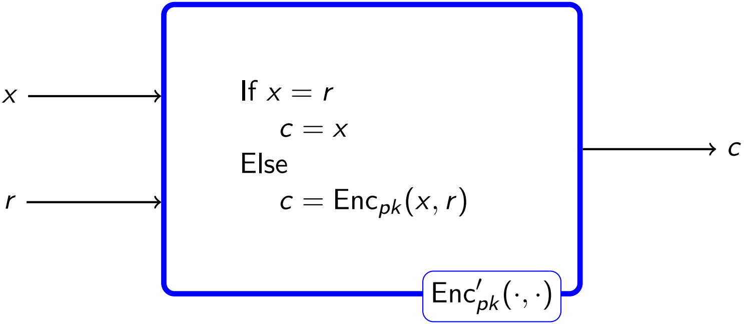

Encrypting randomness dependent messages is a bad idea

A simple example: using message as randomness

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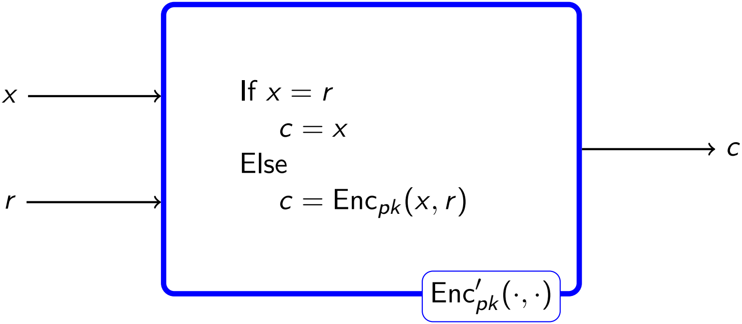

$x \mapsto \mathsf{Enc}'_{pk}(x, x)$ is **not** one-way

Message-dependent randomness

- $x \mapsto \mathsf{Enc}_{pk}(x, x)$ is not one-way
- What about

$$x \mapsto \mathsf{Enc}_{pk}(x, h(x))?$$

This approach is doomed to fail

Theorem ([GMR01])

There is no black-box construction of injective trapdoor functions from IND-CPA secure cryptosystems

Random oracles break message dependency

If Enc is IND-CPA secure, and h is a RO, then

- $x \mapsto \mathrm{Enc}(x, h(x))$ is a one-way trapdoor function [BHSV98]
- $x \mapsto \mathrm{Enc}(x, h(pk, x))$ is deterministic encryption [BBO07]

Dependencies between messages and randomness

- $x \mapsto \mathsf{Enc}(x, x)$ may not be one-way
- $x \mapsto \mathsf{Enc}(x, h(x))$ is one-way when h is a RO
- What if h is a some other function?

Main result

If:

- Enc is *lossy encryption*
- h is a pairwise independent hash function

Then:

$$x \mapsto \mathrm{Enc}(x, h(x)) \text{ is an injective trapdoor function}$$

Main result

If:

- Enc is *lossy encryption*
- h is a pairwise independent hash function
- Message space is larger than the randomness space

Then:

$x \mapsto \mathsf{Enc}(x, h(x))$ is an injective trapdoor function

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Lossy Cryptographic Primitives

- Lossy primitives have two types of public-keys
  - **Injective keys** - these allow decryption / inversion
  - **Lossy keys** - these *statistically* lose information about the message / input
- The two types of keys are computationally indistinguishable

Lossy Encryption

[GOS06, PW08, PVW08, KN08, BHY09]

$$G(1^\lambda, mode), E(pk, m, r), D(sk, c)$$

Correctness:

For all m, r

$$D(E(pk_I, m, r)) = m$$

Lossiness:

For all m_0, m_1

$$\{E(pk_L, m_0, r)\} \approx^s \{E(pk_L, m_1, r)\}$$

Indistinguishability

$$\{pk_I : pk_I \leftarrow G(1^\lambda, Injective)\} \approx^c \{pk_L : pk_L \leftarrow G(1^\lambda, Lossy)\}$$

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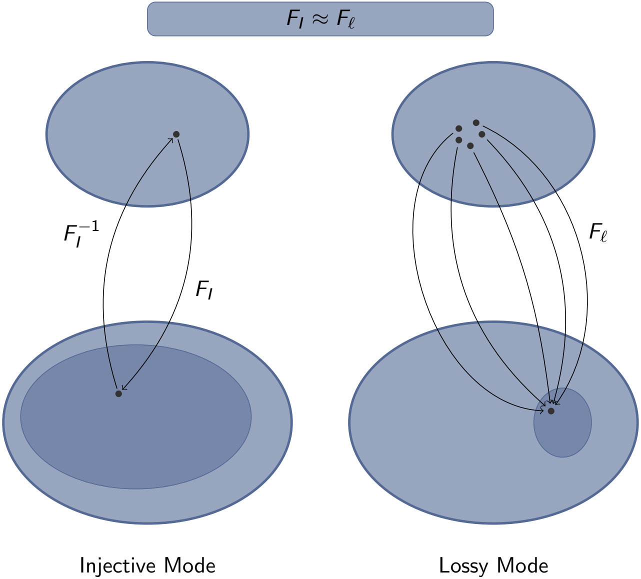
Notice: Indistinguishability + Lossiness $\Longrightarrow$ IND-CPA security

Lossy Trapdoor Functions [PW08]



Lossy Trapdoor Functions in Detail

$$(s, t) \longleftarrow G_{LTDF}(1^\lambda, inj)$$

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Trapdoor:

$$F^{-1}(t, F(s, x)) = x$$

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Lossiness:

$$|\operatorname{im} F(s, \cdot)| \leq 2^r$$

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Trapdoor:
$$F^{-1}(t, F(s, x)) = x$$

Lossiness:
$$|\operatorname{im} F(s, \cdot)| \leq 2^{r}$$

The first outputs of $G_{LTDF}(1^{\lambda}, inj)$, and $G_{LTDF}(1^{\lambda}, lossy)$ are computationally indistinguishable

Constructions of LTFs

- DDH,LWE [PW08]
- DCR [RS08, BFO08]
- D-Linear, QR [FGK$^+$10]
- Φ-Hiding [KOS10]
- EDDH [HO12]

Implications of LTFs

- IND-CCA encryption (also IND-CPA,CRHFs,OT,PRGs) [PW08]
- Deterministic Encryption [BFO08]
- Correlated Product Security [RS09, MY09]
- Replace RO in RSA-OAEP [KOS10]
- Leaky Pseudo-entropy Functions [BHK11]

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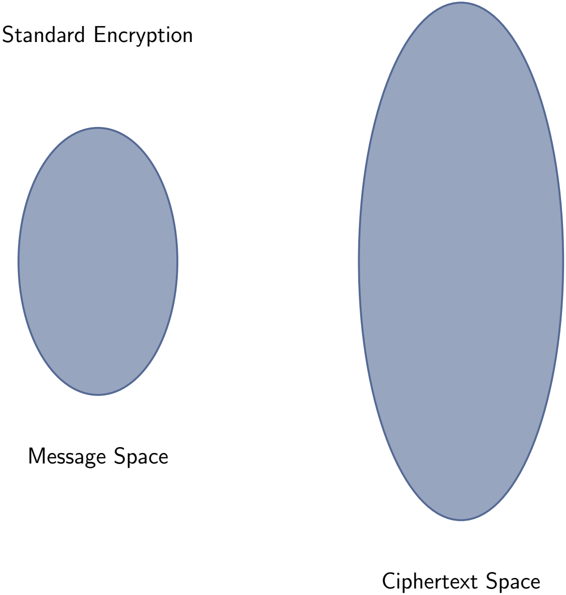
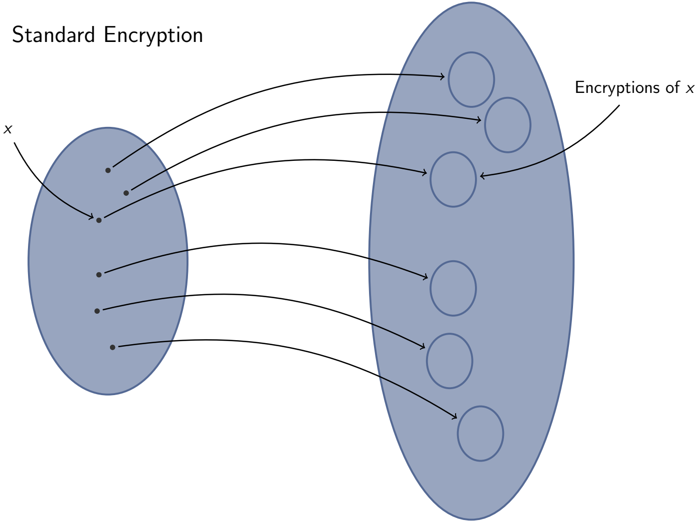


Lossy Encryption



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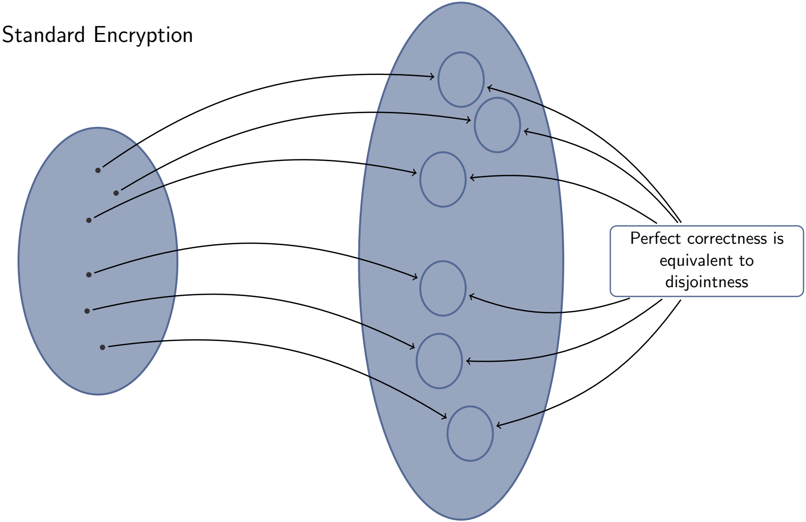
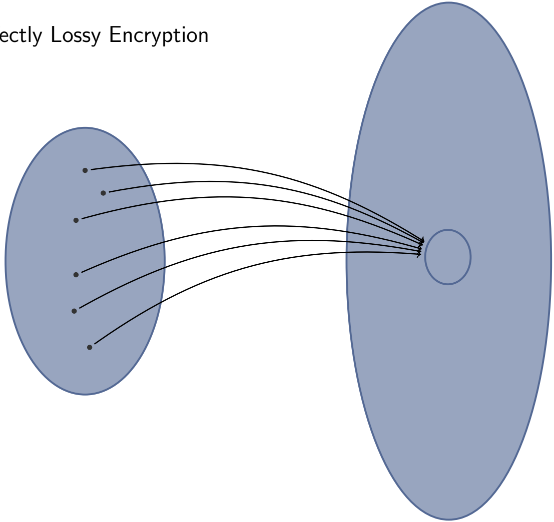


Lossy Encryption

Perfectly Lossy Encryption



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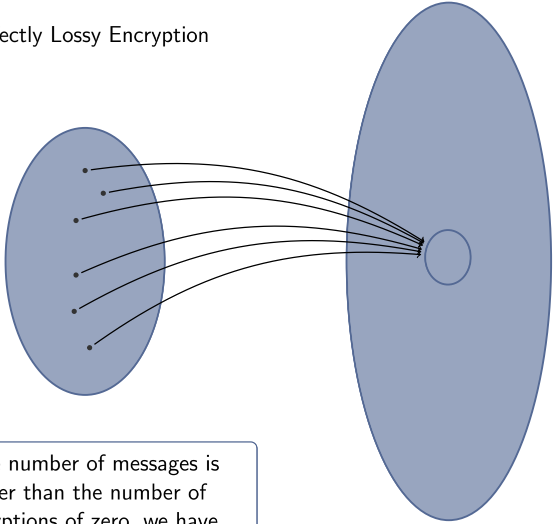

If the number of messages is larger than the number of encryptions of zero, we have lossiness

Perfectly Lossy Encryption Implies LTFs

A simple warmup

- Suppose
  - Enc is a *perfectly lossy encryption*.
  - $|\mathcal{M}| > |\mathcal{R}|$ (|Message Space| > |Randomness Space|)
- Define:

$$F_{pk}(x) = \mathsf{Enc}_{pk}(x, 0)$$

Perfectly Lossy Encryption Implies LTFs

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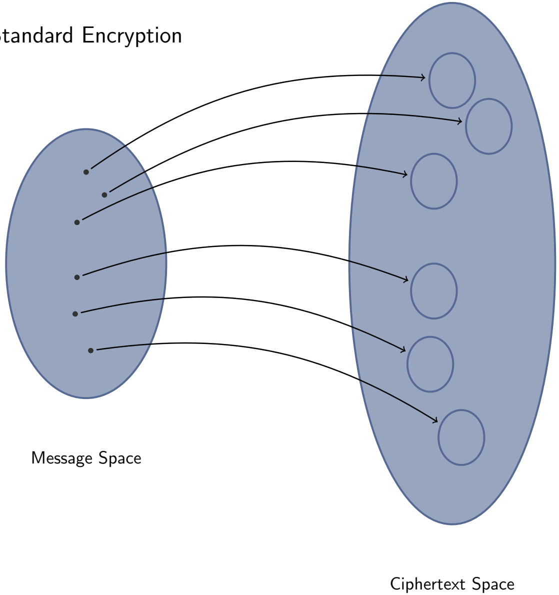
- Suppose
  - Enc is a *perfectly lossy encryption*.
  - $|\mathcal{M}| > |\mathcal{R}|$ ($|$Message Space$| > |$Randomness Space$|$)
- Define:

$$F_{pk}(x) = \mathrm{Enc}_{pk}(x, 0)$$

  Then $F_{pk}(x)$ is a lossy trapdoor function.
- **Proof:**
  In lossy mode, the image of F is bounded by $|\mathcal{R}| < |\mathcal{M}|$.
  Injective and lossy modes are indistinguishable because Enc is a lossy encryption.

Lossy Encryption

Standard Encryption



Lossy Encryption

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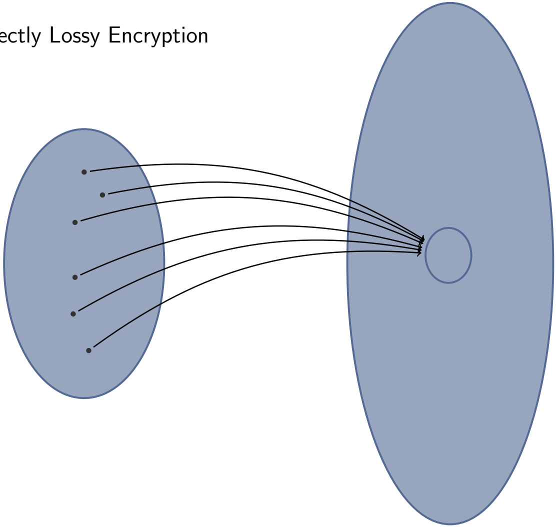
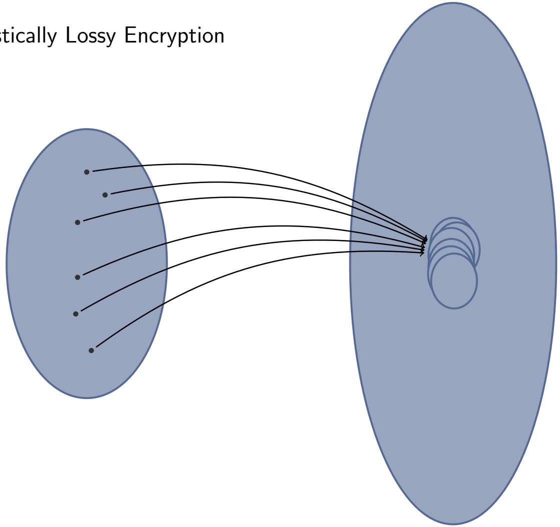


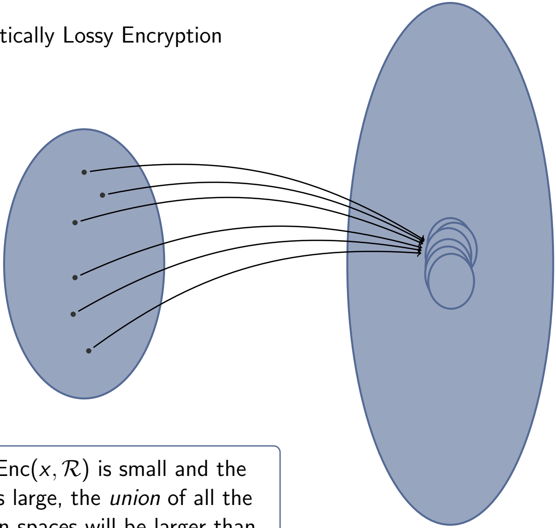
Lossy Encryption

Statistically Lossy Encryption



Lossy Encryption

Statistically Lossy Encryption



Even if $\mathrm{Enc}(x, \mathcal{R})$ is small and the overlap is large, the *union* of all the encryption spaces will be larger than $|\mathcal{M}|$, so the previous argument fails.

Lossy Trapdoor Functions from Lossy Encryption

Main result

- Suppose
  - Enc is a *lossy encryption*.
  - The plaintext space, $\mathcal{M}$ is larger than the randomness space $\mathcal{R}$.
- Define: $F_{pk}(x) = \mathsf{Enc}_{pk}(x, h(x))$
  where h is a pairwise independent hash function. Then $F_{pk}(x)$ is a lossy trapdoor function.

Lossy Trapdoor Functions from Lossy Encryption

- **Proof Sketch:**
  We must show that in lossy mode, with high probability over the choice of h, the size of $\left|\bigcup_{x \in \mathcal{M}} \mathsf{Enc}_{pk}(x, h(x))\right| < |\mathcal{M}|$.
  Let $C_0 = \mathsf{Enc}_{pk}(0, \mathcal{R})$ (the set of encryptions of 0).
- In lossy mode, with high probability over x, $\mathsf{Enc}_{pk}(x, h(x)) \in C_0$.
- Expected number of points $F_{pk}(x) \in C_0$ is large.
- Pairwise independence shows variance is small.
- With high probability most of the evaluations $F_{pk}(x)$ lie in the small space C_0.

Lossy Trapdoor Functions from Lossy Encryption

Consequences

- **Main Result:** Lossy encryption with plaintexts at least one bit longer than the randomness implies LTFs.
- Lossy Encryption is equivalent to statistically sender private 1-2-OT, so statistically hiding OT with long messages implies lossy trapdoor functions and hence injective trapdoor functions.
- The primary open question is whether we can relax the requirement on plaintext length.

Comparison to Non-Lossy Case

- [BHSV98]: when Enc is an IND-CPA secure cryptosystem, and h is a random oracle, $F_{pk}(x) = \mathsf{Enc}_{pk}(x, h(x))$ is an injective trapdoor function.
- [BBO07]: when Enc is an IND-CPA secure cryptosystem, and h is a random oracle, $F_{pk}(x) = \mathsf{Enc}_{pk}(x, h(x||pk))$ is deterministic encryption.
- Our results do *not* require a random oracle.

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Randomness dependent message security

See also [BCPT13]

$$pk \xleftarrow{\$} \mathsf{Gen}(1^\lambda)$$

$$\vec{r} = (r_1, \ldots, r_n) \xleftarrow{\$} \mathsf{coins}(\mathsf{Enc})$$

$$(f_1, \ldots, f_n) \xleftarrow{\$} \mathcal{A}_1(pk)$$

$$\vec{c} = (\mathsf{Enc}(pk, f_1(\vec{r}), r_1), \ldots, \mathsf{Enc}(pk, f_n(\vec{r}), r_n))$$

$$b \leftarrow A_2(\vec{c})$$

Real

Randomness dependent message security

See also [BCPT13]

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$$(f_1, \ldots, f_n) \xleftarrow{\$} \mathcal{A}_1(pk)$$

$$\boxed{\vec{c} = (\mathsf{Enc}(pk, 0, r_1), \ldots, \mathsf{Enc}(pk, 0, r_n))}$$

$$b \leftarrow A_2(\vec{c})$$

Ideal

Randomness dependent message security

See also [BCPT13]

$pk \xleftarrow{\$} \mathsf{Gen}(1^\lambda)$

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$(f_1, \ldots, f_n) \xleftarrow{\$} \mathcal{A}_1(pk)$

$\boxed{\vec{c} = (\mathsf{Enc}(pk, 0, r_1), \ldots, \mathsf{Enc}(pk, 0, r_n))}$

$b \leftarrow A_2(\vec{c})$

Parallels KDM security [BRS03, BHHO08, HU08, ACPS09]

Randomness Circular Security

Definition

A cryptosystem is Randomness Circular Secure if

$$\{pk, \mathrm{Enc}(pk, r_2, r_1), \mathrm{Enc}(pk, r_3, r_2), \ldots, \mathrm{Enc}(pk, r_n, r_{n-1}), \mathrm{Enc}(pk, r_1, r_n)\}$$
$$\approx_c$$
$$\{pk, \mathrm{Enc}(pk, 0, r_1), ..., \mathrm{Enc}(pk, 0, r_n)\}$$

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$$\approx_c$$
$$\{pk, \mathsf{Enc}(pk, 0, r_1), ..., \mathsf{Enc}(pk, 0, r_n)\}$$

Similar to (key) circular security [CL01, BRS03, BHHO08]

RCIRC One-wayness

Definition

A cryptosystem is RCIRC-one-way if the map

$$(r_1, \ldots, r_n) \mapsto (\mathsf{Enc}(pk, r_2, r_1), \ldots, \mathsf{Enc}(pk, r_1, \ldots, r_n))$$

is one-way

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Implies one-way trapdoor functions

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Conclusions

- Lossy Encryption with long plaintexts implies LTFs
- OT with long messages implies injective trapdoor functions

Open Problems

- Does Lossy Encryption imply LTFs?
  i.e. can we drop the restriction on plaintext length?

Thanks

Thanks!

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