Review of the book

"Proofs from THE BOOK, 4^{th} Edition"

by M. Aigner and G. Ziegler

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1 What is this book about?

The Book, as promulgated by Paul Erdős, is God's collection of the most elegant proofs of any and all mathematical theorems, including those still to be discovered. Erdős was undoubtedly one of the most eccentric and prolific mathematicians of the 20th century, having co-authored more than 1400 papers. His belief in the inner aesthetics of mathematics was congruent to and influenced by G.H. Hardy's own belief as expressed in the often quoted phrase: "...there is no lasting place for ugly mathematics". A belief which, judging by the long list of contributers listed in the preface, is held by many mathematicians.

In *Proofs from THE BOOK*, Martin Aigner and Günter M. Ziegler attempt to gather together a collection of proofs which, in their opinion, should be included in *The Book*. The project was actually begun in collaboration with Erdős before his death in 1996; therefore, at least some of the proofs are known to have met with his approval. Although it is not clear to what extent the majority of proofs presented here are generally considered *"elegant"* by the mathematics community at large, or to what extent they reflect a more pedestrian *"beauty is in the eye of the beholder"* bias of the authors, the mere fact that *Proofs from THE BOOK* is in its fourth edition lends credence to claims of widespread acclaim and continuing popularity.

The book is divided into five sections each covering one major area of mathematics:

- Number Theory
- Geometry
- Analysis
- Combinatorics
- Graph Theory

whereby, each section contains eight chapters dealing with a particular subtopic. At the end of each chapter there is a short list of references to the literature for helping the curious to continue exploring on their own. To be clear, this is not a book for learning mathematics. No attempt has been made to be comprehensive, instructive or constructive, neither does *Proofs from THE BOOK* take an axiomatic approach, rather each section is a collection of loosely related theorems and their proofs.

Whether or not the reader's own favorite theorem is represented is not a question of its importance, but seems to be a question of the authors' personal taste. It is at this point, that a more detailed explanation regarding the selection process used by the authors and their collaborators would have been most useful. As it stands, the reader is left with the impression that this slim book of less than 300 pages either contains a purely random selection of theorems, or the beauty so highly prized by mathematicians is actually quite rare in pure mathematics.

To be fair, the authors are constrained by other criterion apart from sheer elegance. These include the length of the proof and, its accessibility to readers whose background includes at the most a few semesters of mathematics at the undergraduate level. Without the latter criterion the book would not be of interest to general audiences, though it does mean you will not find discussed herein famous topics which haunt the laymen, such as Fermat's Last Theorem, or Gödel's Incompleteness Theorem, as the proofs of such theorems require advanced, graduate level concepts. The former criterion also means we are presented with a proof of the five coloring problem in chapter 34, but not its more famous cousin, the four coloring problem, as its proof is much too long.

What the *Proofs from THE BOOK* does provide is a glimpse into the mathematical mind. Browsing through the proofs one gets a sense of the rich creative process involved in proving theorems. Although the book illustrates many standard techniques like deduction or proof-by-contradiction, it also demonstrates how these techniques have to be combined with insightful assumptions and bold leaps of the imagination in order to reach the goal. When students first encounter Cantor's well known diagonalization method for proving that the real numbers are not countable, they often scratch their heads and wonder how he ever thought of that. Well, it seems the authors are just as stumped and describe it as nothing less than "a stroke of genius". Perhaps that is also the best way to describe what an elegant proof is, namely "a stroke of genius".

2 What is the book like?

Proofs from THE BOOK is written in a relaxed style which can be best described as a blend between a university level textbook and an article from *Scientific American*. The authors have done a commendable job of editing the copious inputs they received into a cohesive text exhibiting a consistent, fluent style across the various topics. It follows the familiar theorem-proof format, interspersed with prose providing context information for the theorems as well as occasional comments on why the proof should be considered for inclusion in *The Book*.

Unfortunately, it is exactly this context where the book is at its weakest. In chapter five for example, the authors claim that no less than 196 independent proofs for the law of quadratic reciprocity have been cataloged; yet they do not explain why this curious theorem, which is well known in the cryptographic community, is worthy of such attention. Naturally, any explanation would not recourse to a list of practical applications, since, as the authors have made clear, and Erdős would approve, the purpose of *The Book* is pure mathematics itself; however, it would be interesting for the non-mathematician to know why certain results so pique a mathematicians curiosity that they are proved over and over again. Is it a question of the theorem's broad reach or is it simply a highly intellectual glass beads game à la Hermann Hesse?

On the other hand, there are other examples of theorems for which no context is provided, but for which background information is readily available from other sources. Buffon's needle problem for instance, is presented in chapter 24 as a problem in pure mathematics; however, a simple Internet search turns up the explanation that Buffon's needle is related to a game of chance played in 18^{th} century France; hence like counting cards in black-jack, one could have made money knowing the solution. In other words, despite the numerology surrounding the appearance of π in the result, this theorem has a rather prosaic background.

Finally a note on the book as such. The hardback edition is abundantly illustrated throughout with colorful drawings by Karl H. Hofmann, computer generated graphics and reproductions of paintings and photos of well known mathematicians (Though I suspect Ms. Nilli was bit older than she looks in her picture when she disproved Borsuk's conjecture...). The book has a nice, clear typeset printed on pure white paper with a font that is easy on the eyes and, for a mathematics book, enough white space so as not to be too intimidating.

3 Would you recommend this book?

If *The Book*, as understood by Erdős, is compared with a *Form* in Platonic philosophy, then *Proofs from THE BOOK* is a mere *shadow*. It gives us a flavor of what *The Book* in its pure form would be like, but, just as the authors often present several proofs for a given theorem, seemingly unsure as to which is the most elegant, the reader is also left wondering if the concept of beautiful mathematics can be adequately delineated. This book makes no attempt to offer the reader a conceptual framework for deciding whether or not a proof is beautiful, relying instead on numerous examples of short elegant proofs to convey the notions behind the concept of elegance. In the end though, this is not quite satisfactory. Surely, long convoluted arguments are ugly, but are long proofs inherently inelegant? Do the authors truly expect results like the proof of Fermat's Last Theorem or the Four Color Problem to eventually fit into the margins of their book? If not, then a deeper discussion of what does or does not constitute beauty in mathematics is required.

Quibbles aside, *Proofs from THE BOOK* is highly recommendable, for unlike many popularizations of science and mathematics, it delves into real theorems not muddy metaphors or inconsistent analogies. To be sure, the book is aimed at the educated layman or amateur mathematician, whereby, the word "layman" should be qualified to mean anyone who has had a few years of college level math. While some of the sections are accessible with only a high school background, this, as noted in the preface, is not the main audience; hence, anyone approaching this book should be prepared for pages, often rather dense, of advanced mathematical notation and reasoning. In the end, the reader will be rewarded for his/her effort by a deeper understanding of some intriguing theorems in pure mathematics as well as an understanding of how pure mathematicians approach their main line of work, i.e., proving theorems.

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