

FHE-Based Bootstrapping of Designated-Prover NIZK

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Abstract. We present a novel tree-based technique that can convert any designated-prover NIZK proof system (DP-NIZK) which maintains zero-knowledge only for single statement, into one that allows to prove an unlimited number of statements in ZK, while maintaining all parameters succinct. Our transformation requires leveled fully-homomorphic encryption. We note that single-statement DP-NIZK can be constructed from any one-way function. We also observe a two-way derivation between DP-NIZK and attribute-based signatures (ABS), and as a result derive now constructions of ABS and homomorphic signatures (HS). Our construction improves upon the prior construction of lattice-based DP-NIZK by Kim and Wu (Crypto 2018) since we only require leveled FHE as opposed to HS (which also translates to improved LWE parameters when instantiated). Alternatively, the recent construction of NIZK without preprocessing from either circular-secure FHE (Canetti et al., STOC 2019) or polynomial Learning with Errors (Peikert and Shiehian, Crypto 2019) could be used to obtain a similar final statement. Nevertheless, we note that our statement is formally incomparable to these works (since leveled FHE is not known to imply circular secure FHE or the hardness of LWE). We view this as evidence for the potential in our technique, which we hope can find additional applications in future works.

1 Introduction

In non-interactive zero-knowledge proof systems for NP (NIZK) [BFM88], a prover can provide a non-interactive proof of the validity of an NP statement (efficiently, using a witness), that convinces a verifier, without revealing any information about the witness or anything other than the validity of the statement. This is not possible to achieve in the plain model, and therefore usually

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some common setup is considered, in particular it is often assumed that an honestly generated common reference string (CRS) is accessible to both the prover and the verifier [FLS90]. In this work we consider proof systems with statistical soundness and computational zero-knowledge.

NIZK with Preprocessing. In some cases it suffices to consider a relaxed notion, *NIZK with preprocessing* [SMP87], where the trusted party that generates the CRS also produces additional secret information either for the prover or for the verifier or for both. As pointed out by Kim and Wu [KW18], multi-theorem preprocessing NIZK could replace plain NIZK in a number of applications, e.g. for achieving MPC with low round complexity.

In the case of secret information for the prover, known as *designated-prover* NIZK or DP-NIZK, the prover’s key should be kept secret in order to maintain zero-knowledge. In the mirror case of *designated-verifier* NIZK (DV-NIZK), the verifier’s secret is for the purposes of securing the soundness. In both cases, the preprocessing might make the CRS *non-reusable*. That is, if the same secret key of the prover (resp. verifier) is used in the proofs of multiple statements then ZK (resp. soundness) might not hold for all of these statements. Therefore, we make the distinction between single-theorem and multi-theorem NIZK in the preprocessing model. We note that throughout this introduction, writing DP/DV-NIZK refers by default to the multi-theorem version.

The seminal work of [FLS90] shows, among other things, how to transform any single-statement NIZK proof system into a multi-statement NIZK. [KNYY19] recently showed that a similar bootstrapping strategy works also in the designated-verifier model. As pointed out by [KW18], the transformation fails to work in the designated-prover model since it critically relies on the fact that the prover algorithm is publicly computable. In this work we focus on multi-statement DP-NIZK.

1.1 Our Results

We present a new technique for bootstrapping DP-NIZK from single-theorem to multi-theorem, using leveled fully homomorphic encryption (FHE) as a building block. We recall that leveled FHE schemes are ones that allow to evaluate depth d circuits, for any (polynomially bounded) d specified at key generation time. We start by noticing that single-theorem DP-NIZK can be constructed straightforwardly from any one-way function using garbled circuits and commitment schemes (we did not find this simple construction in the literature). We then apply a succinctness transformation similar to that proposed by Gentry et al. [Gen09, GGI⁺15] to shrink the CRS size and make it independent of the complexity of the statement that needs to be proven. This transformation uses (leveled) FHE. Finally, as our main contribution, we present a tree based construction which transforms single-theorem succinct DP-NIZK into multi-theorem (succinct) DP-NIZK, essentially by committing to an implicit tree of CRS values, each of which is used to prove the validity of its children. We provide more information and a technical overview in Section 1.3 below.

In addition, in this work, we observe a two-way implication between DP-NIZK and the notion of attribute-based signatures (ABS) [MPR11, BF14, BZ14, BGI13], assuming one-way functions exist. Combining this new observation with the known connection between ABS and HS [Tsa17], we get a construction of homomorphic signatures from leveled FHE. We note the parameters of the obtained HS scheme are fairly unfavorable, in particular the length of the signature grows with the size of the evaluated function. However, this schemes has the so-called context-hiding property. Such HS schemes suffice for some applications (not surprisingly, the [KW18] construction of DP-NIZK is an example of such a application). See Section 1.4 below.

1.2 Our Parameters and Assumptions Compared to Prior Work

Comparison with DP-NIZK Constructions from LWE. [KW18] presented a construction of DP-NIZK using homomorphic signatures for \mathbf{NC}^1 as building block. Such HS schemes were constructed under the learning with errors (LWE) assumption [Reg05] by Gorbunov, Vaikuntanathan and Wee [GVW15]. Comparing our result with their work, we point out that our techniques are very different. Succinctness and bootstrapping that play a central role in our construction, do not appear to be a component of the [KW18] construction. In terms of assumptions, we require leveled FHE and they require homomorphic signatures. The two assumptions are formally incomparable, but when instantiating concretely with LWE, our construction is favorable in terms of the required assumption. Leveled FHE can be constructed based on the hardness of LWE, with a fixed polynomial modulus-to-noise ratio, and with parameters that grow moderately with d [BV14]. The modulus-to-noise ratio (when measured as a function of the dimension of the LWE problem) effectively determines the hardness of the LWE instance at hand. The smaller the ratio is, the harder the problem becomes, and the better approximation to worst-case lattice problems one will be able to achieve if the assumption is broken. In terms of parameter growth, the only parameter effected by d is the public key, which grows linearly with d . In contrast to FHE, it is not known how to *bootstrap* homomorphic signatures. Bootstrapping allows the modulus-to-noise ratio to be fixed, regardless of the evaluation depth. Therefore, since [KW18] requires the use of HS rather than FHE, they require modulus to noise ratio of $\text{poly}(s)$, where s is the size of the verification circuit for the \mathbf{NP} relation for which proofs are provided. This is worse than the parameters presented in this work.

We should also compare our construction to the recent constructions of NIZK without preprocessing by Canetti et al. [CCH⁺19] and by Peikert and Shiehian [PS19]. The former constructs NIZK from any *circular secure* FHE scheme. That is, one that can securely encrypt its own secret key. This is not known to be implied by LWE, but it is an assumption that is fairly common in the FHE literature. The latter constructs NIZK from LWE with fixed polynomial modulus-to-noise ratio. Their construction uses LWE-based leveled FHE as building block, but then uses specific properties of the LWE-based scheme so their construction is not generic. Formally, none of these constructions are

implied by generic leveled FHE, which suggests that our techniques have a novel aspect that hopefully can serve as stepping stone for future contributions.

Other DP-NIZK Constructions. Katsumata et al. [KNYY19] showed how to construct DP-NIZK (with computational soundness) based on a (new) assumption on groups with bilinear maps, however a later work by these authors [KNYY20] subsumed that result and showed how to remove the preprocessing and remain with essentially the same properties.

1.3 Technical Overview

As we outlined above, our construction has three components:

1. A single-theorem DP-NIZK construction from any one-way function via garbled circuits and commitments.
2. A succinctness transformation from single-theorem DP-NIZK to succinct single-theorem DP-NIZK. This is similar to the succinctness transformations in [Gen09, GGI⁺15].
3. Tree-based bootstrapping from succinct single-theorem NIZK to (succinct) multi-theorem DP-NIZK.

In what follows, we describe each of these components in more detail. We consider an **NP** language L and we let V be the verifier for an **NP** relation of L . That is, V is a polynomial time algorithm s.t. $V(x, w) = 1$ if and only if w is a valid witness for $x \in L$. We slightly overload the notation and also use V to denote the circuit that implements the algorithm V on instances x of length n , where n is clear from the context.

Single-Theorem DP-NIZK from OWF. We create a DP-NIZK scheme for instances of size n with respect to the language L . The DP-NIZK setup generates a common reference string crs and prover secret key k_P as follows. We start by generating a garbled circuit G of the circuit V . We then commit to each of the labels of the garbled circuit, let $c_{i,b}$ be a commitment to the label $\ell_{i,b}$. The garbled circuit G and the committed labels are placed in crs , and the openings of all commitments are provided as the secret k_P . In order to prove that $x \in L$ for some instance x , a prover P with witness w simply opens the commitments to the labels corresponding to (x, w) . The verifier executes the garbled circuit and verifies that the output is indeed 1. One minor subtlety is that the verifier needs to be convinced that the prover indeed opened to the correct x , but needs to know nothing about w . This is achieved by providing the labels corresponding to x in the “correct” order, i.e. $c_{i,0}$ followed by $c_{i,1}$, but for the w labels $c_{i,0}$ and $c_{i,1}$ will be randomly permuted.

Remark 1. As was pointed out to us by TCC 2020 reviewers, there is an alternative way to obtain single-theorem DP-NIZK from OWF by instantiating the hidden-bit model [FLS90]. The hidden bit-model requires that the CRS constitutes a commitment to a sequence of bits drawn (by a trusted party) from a

certain distribution, which can be opened by the prover. This can be instantiated for designated prover straightforwardly by having in the CRS commitments to the hidden bits, and giving the openings of the commitments to the prover.

This alternative has the advantage of not requiring use of garbled circuits, but it has the drawback of being designated for a specific **NP** complete language such as hamiltonicity, or variants of SAT [KP95]. Although in a different context, [Dam92] also considers the goal of constructing proofs directly for arbitrary circuits. In contrast, the garbled circuit approach can apply directly to any witness relation and does not require an **NP** reduction to be used.

Succinctness Transformation. The idea here is to use (leveled) FHE as follows. In the setup process, generate crs^0, k_P^0 for a non-compact scheme, and also generate a key pair for the FHE scheme (hpk, hsk). Place hpk, crs^0 and a commitment c for hsk in the new CRS and hsk, k_P^0 and the opening for the commitment c in the new k_P . Now, to prove that $x \in L$ using w , encrypt w using the FHE, let ct_w be the encryption. Then consider the ciphertext ct which is the homomorphic evaluation of $V(x, \cdot)$ on the ciphertext ct_w . The prover will use k_P^0 to prove that when decrypting ct with the hsk committed to by the commitment c , the outcome is 1. The verifier will calculate ct locally using homomorphic evaluation, and then will verify the proof using crs^0 . This guarantees that soundness holds, up to subtleties like ensuring that any ciphertext ct_w , even dishonestly generated, corresponds to some encrypted value. We note that soundness for DP-NIZK is easier to show than for NIZK without preprocessing as in [Gen09, GGI⁺15] since the homomorphic encryption keys, and commitments thereof, are guaranteed to be honestly generated.

In terms of succinctness, we only use the underlying scheme to prove statements about the decryption circuit of the FHE scheme, which is independent of the statement length n , and therefore we would hope that the complexity of V does not play a role in the parameters of the new scheme. This is not entirely correct since in leveled FHE, the length of hpk can depend on the depth of V . However, we note that hpk is reusable, and we can generate many instances of the proof system with the same hpk . Thus, the parameters contain a part which may not be succinct but is reusable, and another part that is succinct but possibly not reusable. This will suffice for our purposes as we see below.

Tree-Based Bootstrapping. The basic idea is to implicitly generate exponentially many (in the security parameter) single-theorem (crs, k_P) values, so that the prover can use a fresh value for each new theorem. Of course the verifier will need a way to retrieve the correct crs and verify that it is indeed one of those implicit crs values and not some value maliciously chosen by a dishonest prover. To resolve this issue, we generate additional instances of the single-theorem DP-NIZK and use them to prove that the crs values were generated honestly according to some predetermined (pseudo)randomness that is given via a PRF.

In more detail, we consider a depth- λ binary tree, where each node is associated with an independent instance of the single-theorem DP-NIZK, and all of those instances are implicitly determined by a PRF seed that generates the

randomness for the DP-NIZK setup algorithm. Every intermediate node is used in order to prove the (single) statement that the CRS of its two children were generated honestly according to the PRF seed, and the CRS of the tree-root is given as part of the CRS of the multi-theorem DP-NIZK. To prove a statement, the prover randomly chooses one of the tree leaves and uses the corresponding CRS to generate the proof for the statement. It then provides, in addition, all of the proofs on the path from the tree root to this leaf, as an evidence that this leaf indeed appears in the predetermined tree.

Note that the setup algorithm of any node is encoded into the NP relation that is proved by its parent node, and a non-efficient setup might cause a blow-up that is exponential with the tree depth. This is where we crucially use the decomposability of the setup algorithm that was discussed when we described the succinctness transformation. In the tree construction, we will generate the reusable-but-not-succinct part of the single-statement CRS once for all of the nodes in a given level of the tree, and then each node will be associated with a new instance of the succinct-but-not-reusable part of the single-statement CRS.

The zero-knowledge comes from the zero-knowledge of the single-statement DP-NIZK (as long as the prover does not choose the same leaf more than once) and from the pseudorandomness of the PRF seed. In fact, we have to generate a fresh PRF seed for every level of the tree, and to use the pseudorandomness of the seed of the i th level to claim that the single-statement DP-NIZK instances of the i th level are zero-knowledge. We then can claim that the zero-knowledge of the i th level instances guarantees that the PRF seed of the $i + 1$ th level remains secret since it is only used as a witness, and the proof proceed via 2λ similar hybrids.

If we only cared about zero-knowledge, we could let the prover sample the PRF seeds on its own (or even to use real randomness instead of PRF outputs). But such scheme would not be sound, since the prover can possibly sample a bad CRS of the single-statement DP-NIZK for which soundness does not hold. To resolve this issue, during setup we sample a random string r along with the PRF seeds for all of the tree levels, and publish r and commitments to the seeds. The prover is then forced to use as “randomness” for the single-statement DP-NIZK setup the PRF outputs XORed with the truly random string r , where we enforce this as part of the NP relation that is verified. That is, we require that the witnesses of all of the proofs along the tree will include the proper decommitments to the PRF seeds. With this approach the PRF seeds remain hidden from the verifier due to the hiding of the commitments, so we don’t compromise zero-knowledge, and in addition the *truly* random string r restricts the prover to use as “randomness” for the single-statement DP-NIZK setup only strings which the marginal distribution of each of them independently is uniform. We therefore need the underlying single-statement DP-NIZK to be sound for any set of 2^λ CRS values that were sampled with “randomness” that is randomized via the same uniform string r . To obtain this, we use the fact that the underlying single-statement DP-NIZK is *statistically* sound and therefore its soundness can be amplified via parallel repetition λ times.

1.4 DP-NIZK, Attribute-Based Signatures and Homomorphic Signatures

An attribute-based signatures scheme (ABS, [MPR11, BGI13, BF14, BZ14]) is a digital signature scheme that supports multiple keys with varying permissions, where signatures do not reveal information about the permissions of the signing key that was used. A homomorphic signature (HS, [CJL09, BFKW09, GKCR10, BF11a, BF11b, GVW15]) is a digital signature that supports homomorphic evaluations over the signed message, where evaluated signatures should not reveal information about the message associated with the pre-evaluated signature other than the result of the function that was computed homomorphically.

The relation between ABS, HS and NIZK was studied in various works. [MPR11, BF14, SAH16, SKAH18] show reductions of the form “OWF+NIZK \rightarrow ABS”, [KW18] show that “HS \rightarrow DPNIZK” and [Tsa17] shows that “ABS \leftrightarrow HS” for certain types of ABS and HS. Our new DPNIZK construction can be translated to new ABS and HS constructions as follows.

Attribute-Based Signatures from OWF and DP-NIZK. While we believe that some of the aforementioned constructions [MPR11, BF14, SAH16, SKAH18] of ABS from OWF+NIZK can possibly be initialized from OWF+DPNIZK (and in turn also imply HS from OWF+DPNIZK via [Tsa17]), to the best of our knowledge a statement of the flavor “OWF+DPNIZK \rightarrow ABS” does not explicitly appear in previous literature, so we briefly describe such a reduction now.

The ABS public key and master secret key are an instance of a standard signature scheme. To generate an ABS key for a policy f , generate an instance of DPNIZK and a commitment scheme. Commit to f and use the master secret key to sign (with a standard signature) the DPNIZK CRS and the commitment to f . To sign a message x with a constrained key, provide a DPNIZK proof respective to the instance that appears in the key, proving that “there exists a valid decommitment for some f such that $f(x) = 1$ ”. The commitments scheme and standard signature schemes can be instantiated from one-way functions.

Attribute-Based Signatures from FHE. Applying the transformation which is described above to our DPNIZK construction results in an ABS scheme with the following characteristics:

- *Efficiency.* The size of public parameters and the master key is some $\text{poly}(\lambda)$ and in particular independent of the message and policy space, while keys and signatures grow with the policy size.
- *Unforgeability.* The unforgeability is based on the statistical soundness of DPNIZK, the (possibly statistical) binding of the commitment scheme and the unforgeability of the standard signature scheme. Since any ABS is in particular a standard signature scheme, this is the best possible unforgeability.
- *Policy Privacy.* The privacy is based on the hiding of the commitment and on the zero-knowledge of DPNIZK, which in turn relies on the security of the underlying FHE.

Homomorphic Signatures from FHE. We apply the “ABS \rightarrow HS” transformation of [Tsa17] and derive a single-hop HS scheme with the following characteristics:

- *Efficiency.* The size of public parameters and the master key is some $\text{poly}(\lambda)$ and in particular independent of the message and policy space. However, both post-evaluation and pre-evaluation signatures grow with the function to be computed. That is, when one signs a message they also commit to the maximal size of functions to be homomorphically-computed over it.
- *Unforgeability.* Can be based on any OWF.
- *Context-Hiding.* Relies on the security of the FHE.

DP-NIZK from Attribute-Based Signatures. As mentioned above, the work of [Tsa17, KW18] implies a derivation of the form “ABS \rightarrow DPNIZK”. To simplify and complete the picture, we now briefly describe a direct and simple transformation. In the setup of the DP-NIZK scheme, sample a symmetric key sk and initialize the ABS scheme. If $V(\cdot, \cdot)$ is the verification circuit of the NP relation, then consider the circuit $V'(\cdot, \cdot) = V(\cdot, \text{Dec}_{\text{sk}}(\cdot))$ and generate an ABS key for the policy V' . The secret prover key consists of sk and the ABS key for V' . To prove a statement x with a witness w , consider w' the encryption of w under key sk and provide an ABS signature for the message (x, w') .

2 Preliminaries

2.1 Notations

For $n \in \mathbb{N}$ we let $[n]$ denote the ordered set $\{1, \dots, n\}$. For a bit-string $m \in \{0, 1\}^n$ we let U_m^d denote the universal circuit that takes as input a description of a circuit $f : \{0, 1\}^n \rightarrow \{0, 1\}$ of depth at most d , and outputs $f(m)$. for a bit-string $m \in \{0, 1\}^n$ we let m_i denote the i th bit of m .

2.2 Pseudorandom Function (PRF)

Definition 1. A *Pseudorandom Function (PRF)* is a pair of polynomial-time algorithms $(\text{Setup}, \text{Eval})$ where Setup is randomized and Eval is deterministic, such that for any PPT adversary \mathcal{A} it holds that

$$\left| \Pr \left[\mathcal{A}^{\text{Eval}_k(\cdot)}(1^\lambda) = 1 \right] - \Pr \left[\mathcal{A}^{\mathcal{O}(\cdot)}(1^\lambda) = 1 \right] \right| = \text{negl}(\lambda)$$

where the probability is over $k \leftarrow \text{Setup}(1^\lambda)$ and the coins of \mathcal{A} , and $\mathcal{O}(\cdot)$ is a random function.

2.3 Collision Resistant Hash Function (CRH)

Definition 2. An *efficient function family ensemble* $\mathcal{H} = \{\mathcal{H}_{n,\lambda} : \{0, 1\}^n \rightarrow \{0, 1\}^\lambda\}_{n,\lambda \in \mathbb{N}}$ is a *secure collision-resistant hash (CRH) function family* if for any PPT algorithm \mathcal{A} and any n , for large enough λ it holds that

$$\Pr \left[x \neq y, H(x) = H(y) : H \leftarrow \mathcal{H}_{n,\lambda}, (x, y) \leftarrow \mathcal{A}(1^{\lambda+n}, H) \right] = \text{negl}(\lambda) .$$

2.4 Statistically Binding Equivocable Commitments

Definition 3. A commitment scheme $(\text{Gen}, \text{Commit}, \text{Ver})$ is a tuple of PPT algorithms as follows.

- $\text{Gen}(1^\lambda, 1^n) \rightarrow \text{crs}$ takes as input a security parameter λ and message length n , and outputs a common reference string crs .
- $\text{Commit}(\text{crs}, m) \rightarrow (c, d)$ takes as input a common reference string crs and a message $m \in \{0, 1\}^n$, and outputs a commitment c and decommitment d .
- $\text{Ver}(\text{crs}, c, m, d) \rightarrow \{\text{accept}, \text{reject}\}$ takes as input a common reference string crs , a commitment c , a message m and a decommitment d , and either accepts or rejects.

Correctness. A commitment scheme is correct if for every $m \in \{0, 1\}^n$ it holds that

$$\text{Ver}(\text{crs}, c, m, d) = \text{accept}$$

where $\text{crs} \leftarrow \text{Gen}(1^\lambda, 1^n)$ and $(c, d) \leftarrow \text{Commit}(\text{crs}, m)$.

Statistical Binding. A commitment scheme is statistically binding if for any sufficiently large λ and any n the following holds

$$\Pr_{\text{crs} \leftarrow \text{Gen}(1^\lambda, 1^n)} \left[\begin{array}{c} c := \text{Commit}(\text{crs}, m_0 ; r) \\ \exists(r, m_0, m_1, d) : \text{Ver}(\text{crs}, c, m_1, d) = \text{accept} \\ m_0 \neq m_1 \end{array} \right] = \text{negl}(\lambda) .$$

Hiding. A commitment scheme is hiding if for any sufficiently large λ and any n , for any PPT adversary \mathcal{A} and any pair of messages $m_0, m_1 \in \{0, 1\}^n$ it holds that

$$|\Pr[\mathcal{A}(\text{crs}, c_0) = 1] - \Pr[\mathcal{A}(\text{crs}, c_1) = 1]| = \text{negl}(\lambda)$$

where $\text{crs} \leftarrow \text{Gen}(1^\lambda, 1^n)$, $(c_b, d_b) \leftarrow \text{Commit}(\text{crs}, m_b)$ for $b \in \{0, 1\}$ and the probability is over the coins of Gen , Commit and \mathcal{A} .

Equivocability. A commitment scheme is equivocable if there exists a PPT simulator $\mathcal{S} = (\mathcal{S}^A, \mathcal{S}^B)$ such that for any sufficiently large λ and any n , for any PPT distinguisher Ψ , any pair of messages $m_0, m_1 \in \{0, 1\}^n$ and any $b \in \{0, 1\}$ it holds that

$$|\Pr[\Psi(\text{crs}, c_b, d_b) = 1] - \Pr[\Psi(\text{crs}', c', d'_b) = 1]| = \text{negl}(\lambda)$$

where $\text{crs} \leftarrow \text{Gen}(1^\lambda, 1^n)$, $(c_b, d_b) \leftarrow \text{Commit}(\text{crs}, m_b)$ for $b \in \{0, 1\}$, $(\text{crs}', c', \text{td}_{c'}) \leftarrow \mathcal{S}^A(1^\lambda, 1^n)$ and $d'_b \leftarrow \mathcal{S}^B(\text{crs}', c', \text{td}_{c'}, m_b)$ for $b \in \{0, 1\}$, and the probability is over the coins of Gen , Commit , \mathcal{S} and Ψ .

2.5 Garbled Circuits

Definition 4. A garbling scheme for circuits is a tuple of PPT algorithms (Garble, Eval) with the following syntax.

- $\text{Garble}(1^\lambda, C) \rightarrow (\tilde{C}, \{\text{lab}_{i,b}\}_{i \in [n], b \in \{0,1\}})$ is a probabilistic algorithm that takes as input a security parameter λ and a boolean circuit $C : \{0,1\}^n \rightarrow \{0,1\}$, and outputs a garbled circuit \tilde{C} and $2n$ labels $\{\text{lab}_{i,b}\}_{i \in [n], b \in \{0,1\}}$, where each of the labels is of size $\lambda = \text{poly}(\lambda)$ for some fixed polynomial poly .
- $\text{Eval}(\tilde{C}, \{\text{lab}_i\}_{i \in [n]}) \rightarrow b$ is a deterministic algorithm that takes as input a garbled circuit \tilde{C} and n labels $\{\text{lab}_i\}_{i \in [n]}$, and outputs a bit $b \in \{0,1\}$.

Correctness. The scheme is correct if for every circuit $C : \{0,1\}^n \rightarrow \{0,1\}$, every input $x \in \{0,1\}^n$ and every $(\tilde{C}, \{\text{lab}_{i,b}\}_{i \in [n], b \in \{0,1\}}) \leftarrow \text{Garble}(1^\lambda, C)$, it holds that

$$\text{Eval}(\tilde{C}, \{\text{lab}_{i,x_i}\}_{i \in [n]}) = C(x).$$

Security. The scheme is secure if there exists a PPT simulator \mathcal{S} such that for every circuit $C : \{0,1\}^n \rightarrow \{0,1\}$ and every input $x \in \{0,1\}^n$ it holds that

$$(\tilde{C}, \{\text{lab}_{i,x_i}\}_{i \in [n]}) \equiv_\lambda \mathcal{S}(1^\lambda, 1^{|C|}, C(x)),$$

where $(\tilde{C}, \{\text{lab}_{i,b}\}_{i \in [n], b \in \{0,1\}}) \leftarrow \text{Garble}(1^\lambda, C)$ and \equiv_λ denotes computational indistinguishability with respect to the security parameter λ .

2.6 Homomorphic Encryption

Definition 5. A leveled fully homomorphic encryption scheme FHE is a tuple of PPT algorithms (Keygen, Enc, Eval, Dec) with the following syntax.

- $\text{Keygen}(1^\lambda, 1^d) \rightarrow (\text{pk}, \text{sk})$ is a probabilistic algorithm that takes as input a security parameter λ and depth d , and outputs a public key pk and secret key sk .
- $\text{Enc}(\text{pk}, m) \rightarrow \text{ct}$ is a probabilistic algorithm that takes as input a public key pk and a message $m \in \{0,1\}^*$, and outputs a ciphertext ct .
- $\text{Eval}(\text{ct}, C) \rightarrow \text{ct}'$ is a deterministic algorithm that takes as input a ciphertext ct and a boolean circuit $C : \{0,1\}^* \rightarrow \{0,1\}$, and outputs an evaluated ciphertext ct' .
- $\text{Dec}(\text{sk}, \text{ct}') \rightarrow b$ is a deterministic algorithm that takes as input a secret key sk and an evaluated ciphertext ct' , and outputs a bit $b \in \{0,1\}$.

Correctness. The scheme is correct if for every $n, d \in \mathbb{N}$, every message $m \in \{0, 1\}^n$ and every circuit $C : \{0, 1\}^n \rightarrow \{0, 1\}$ of depth at most d , it hold that

$$\begin{aligned} & Pr[(\text{pk}, \text{sk}) \leftarrow \text{Keygen}(1^\lambda, 1^d) , \\ & \quad \text{ct} \leftarrow \text{Enc}(\text{pk}, m) , \\ & \quad \text{ct}' \leftarrow \text{Eval}(\text{ct}, C) , \\ & \quad \text{Dec}(\text{sk}, \text{ct}') \neq C(m)] = \text{negl}(\lambda) , \end{aligned}$$

where the probability is over the coins of Keygen and Enc .

Security. The scheme is secure if for any PPT adversary \mathcal{A} , any $n \in \mathbb{N}$ and any pair of messages $m_0, m_1 \in \{0, 1\}^n$, it holds that

$$|Pr[\mathcal{A}(\text{Enc}(\text{pk}, m_0)) = 1] - Pr[\mathcal{A}(\text{Enc}(\text{pk}, m_1)) = 1]| = \text{negl}(\lambda)$$

where $\text{pk} \leftarrow \text{Keygen}(1^\lambda, 1^d)$ and the probability is over the coins of Keygen , Enc and \mathcal{A} .

Compactness. The scheme is compact if there exists a polynomial $p = p(\cdot)$ such that for all security parameters λ and all $n, d \in \mathbb{N}$, $m \in \{0, 1\}^n$ and $C : \{0, 1\}^n \rightarrow \{0, 1\}$ of depth at most d , for all $(\text{pk}, \text{sk}) \leftarrow \text{Keygen}(1^\lambda, 1^d)$, the output length of $\text{Eval}(\tilde{m}, C)$ is at most p bits long where $\tilde{m} \leftarrow \text{Enc}(\text{pk}, m)$, and the size of sk is at most p bits.

For our application we need an FHE scheme where the correctness also holds for maliciously chosen ciphertexts. Formally,

Definition 6 (FHE with correctness for all ciphertexts). An FHE scheme has correctness for all ciphertexts if for all $n, d \in \mathbb{N}$, for every string $\text{ct} \in \{0, 1\}^p$ and every circuit $C : \{0, 1\}^n \rightarrow \{0, 1\}$ of depth at most d , it holds that

$$\begin{aligned} & Pr[(\text{pk}, \text{sk}) \leftarrow \text{Keygen}(1^\lambda, 1^d) , \\ & \quad \text{ct}'_C \leftarrow \text{Eval}(\text{ct}, C) , \\ & \quad \text{ct}'_I \leftarrow \text{Eval}(\text{ct}, I) , \\ & \quad \text{Dec}(\text{sk}, \text{ct}'_C) \neq C(\text{Dec}(\text{sk}, \text{ct}'_I))] = \text{negl}(\lambda) , \end{aligned}$$

where I is the identity circuit and the probability is over the coins of Keygen .

We now show that any FHE scheme with standard correctness implies a scheme with correctness for all ciphertexts (which preserves the compactness property).

Lemma 1. Let $\text{FHE} = (\text{Keygen}, \text{Enc}, \text{Eval}, \text{Dec})$ be an FHE scheme with standard correctness and let $\text{PKE} = (\text{Keygen}, \text{Enc}, \text{Dec})$ be a public-key encryption scheme. Then there exists an FHE scheme $\text{FHE}' = (\text{Keygen}', \text{Enc}', \text{Eval}', \text{Dec}')$ with correctness for all ciphertexts.

Proof. Define $\text{FHE}' = (\text{Keygen}', \text{Enc}', \text{Eval}', \text{Dec}')$ as follows:

- $\text{Keygen}'(1^\lambda, 1^d)$: Sample $(\text{hpk}, \text{hsk}) \leftarrow \text{FHE.Keygen}(1^\lambda, 1^d)$ and $(\text{pk}, \text{sk}) \leftarrow \text{PKE.Keygen}(1^\lambda)$, then compute $\tilde{\text{sk}} \leftarrow \text{FHE.Enc}(\text{hpk}, \text{sk})$ and output $(\text{pk}', \text{sk}') := ((\text{hpk}, \text{pk}, \tilde{\text{sk}}), \text{hsk})$ where $d' = \text{poly}(d, \lambda)$ is the maximal depth of C' as defined in Eval' below.
- $\text{Enc}'(\text{pk}', m)$: Compute and output $\text{ct}' := m' \leftarrow \text{PKE.Enc}(\text{pk}, m)$.
- $\text{Eval}'(\text{ct}', C)$: Define the circuit $C_{\text{ct}'}(\circ) := C(\text{PKE.Dec}_\circ(\text{ct}'))$. Compute and output $\text{ct}'' := \text{FHE.Eval}(\tilde{\text{sk}}, C_{\text{ct}'})$.
- $\text{Dec}'(\text{sk}', \text{ct}')$: Output $\text{FHE.Dec}(\text{hsk}, \text{ct}')$.

Fix $n, d \in \mathbb{N}$, a string $\text{ct}' \in \{0, 1\}^p$ and a circuit $C : \{0, 1\}^n \rightarrow \{0, 1\}$ of depth at most d . Consider $(\text{pk}', \text{sk}') \leftarrow \text{Keygen}'(1^\lambda, 1^d)$, $\text{ct}''_I \leftarrow \text{Eval}'(\text{ct}', I)$ and $\text{ct}''_C \leftarrow \text{Eval}'(\text{ct}', C)$, then it holds that

$$\text{ct}''_C = \text{FHE.Eval}(\tilde{\text{sk}}, C_{\text{ct}'})$$

$$\text{ct}''_I = \text{FHE.Eval}(\tilde{\text{sk}}, I_{\text{ct}'})$$

and therefore

$$\begin{aligned} \text{Dec}'(\text{sk}', \text{ct}''_C) &= \text{FHE.Dec}(\text{hsk}, \text{ct}''_C) \\ &= \text{FHE.Dec}(\text{hsk}, \text{FHE.Eval}(\tilde{\text{sk}}, C_{\text{ct}'})) \\ &= C_{\text{ct}'}(\text{sk}) \\ &= C(\text{PKE.Dec}_{\text{sk}}(\text{ct}')) \end{aligned}$$

and

$$\begin{aligned} C(\text{Dec}'(\text{sk}', \text{ct}''_I)) &= C(\text{FHE.Dec}(\text{hsk}, \text{ct}''_I)) \\ &= C(\text{FHE.Dec}(\text{hsk}, \text{FHE.Eval}(\tilde{\text{sk}}, I_{\text{ct}'})) \\ &= C(I_{\text{ct}'}(\text{sk})) \\ &= C(\text{PKE.Dec}_{\text{sk}}(\text{ct}')) . \end{aligned}$$

3 Definitions of Designated-Prover NIZK

Definition 7 (DP-NIZK Proofs). A designated-prover non-interactive zero-knowledge (DP-NIZK) proof Π_{DPNIZK} for an ensemble of **NP** languages $\mathcal{C} \subseteq \{C : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}\}$ (where C is a verification circuit and $\mathcal{L}_C = \{x : \exists w \ C(x, w) = 1\}$ is the **NP** language determined by C) is defined by a tuple of PPT algorithms with the following syntax.

- $\text{Setup}(1^\lambda, \text{params}) \rightarrow (\text{crs}, k_P)$ takes as input the security parameter λ and possibly some parameters params of \mathcal{C} (e.g. the maximal circuit depth), and outputs a common reference string crs and a proving key k_P .
- $\text{Prove}_{\text{crs}}(C, k_P, x, w) \rightarrow \pi$ takes as input a common reference string crs , a circuit $C \in \mathcal{C}$, a proving key k_P , a statement x and a witness w . It outputs a proof π .

- $\text{Verify}_{\text{crs}}(C, x, \pi) \rightarrow \{0, 1\}$ takes as input a common reference string crs , a circuit $C \in \mathcal{C}$, a statement x and a proof π , and either accepts (with output 1) or rejects (with output 0) the proof.

Moreover, Π_{DPNIZK} should satisfy the following properties:

(Perfect) *Completeness.* For all sufficiently large λ , for all circuits $C \in \mathcal{C}$, for all pairs (x, w) for which $C(x, w) = 1$ and for all $(\text{crs}, k_P) \leftarrow \text{Setup}(1^\lambda, \text{params})$, it holds that

$$\Pr[\text{Verify}_{\text{crs}}(C, x, \text{Prove}_{\text{crs}}(C, k_P, x, w)) = 1] = 1 .$$

(Statistical) *Soundness.* For all sufficiently large λ and for all $C \in \mathcal{C}$ it holds that

$$\Pr_{\text{crs} \leftarrow \text{Setup}(1^\lambda, \text{params})}[\exists(x, \pi) : x \notin \mathcal{L}_C \wedge \text{Verify}_{\text{crs}}(C, x, \pi) = 1] = \text{negl}(\lambda) .$$

(Programmable CRS) *Zero-Knowledge.* For all PPT adversaries \mathcal{A} there exists a PPT simulator $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ such that

$$\left| \Pr[\mathcal{A}^{\text{Prove}_{\text{crs}}(\cdot, k_P, \cdot, \cdot)}(\text{crs}) = 1] - \Pr[\mathcal{A}^{\mathcal{O}(\cdot, \text{crs}', \tau, \cdot, \cdot)}(\text{crs}') = 1] \right| = \text{negl}(\lambda) ,$$

where $(\text{crs}, k_P) \leftarrow \text{Setup}(1^\lambda, \text{params})$, $(\text{crs}', \tau) \leftarrow \mathcal{S}_1(1^\lambda, \text{params})$ and

$$\mathcal{O}(C, \text{crs}', \tau, x, w) = \begin{cases} \mathcal{S}_2(C, \text{crs}', \tau, x) & C(x, w) = 1 \\ \perp & \text{o.w.} \end{cases} ,$$

and the probability is over the coins of $\mathcal{A}, \mathcal{S}, \text{Setup}, \text{Prove}$. We also consider the a relaxed notion of single-statement zero knowledge, in which the (programmable CRS) zero-knowledge condition holds only for adversaries \mathcal{A} that make at most a single query to the oracle.

We sometimes require the following additional property.

Efficient Setup. A DP-NIZK proof system is efficient if for all λ there exists a $p = \text{poly}(\lambda)$ such that for all params , the complexity of $\text{Setup}(1^\lambda, \text{params})$ is p (and in particular does not depend on params).

3.1 Single-Statement Global-Setup DP-NIZK Proofs

Definition 8 (Single-Statement Global-Setup DP-NIZK Proofs). A single-statement global-setup DP-NIZK proof Π_{IDPNIZK} for an ensemble of NP languages $\mathcal{C} \subseteq \{C : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}\}$ (where C is a verification circuit and $\mathcal{L}_C = \{x : \exists w \ C(x, w) = 1\}$ is the NP language determined by C) is defined by a tuple of PPT algorithms with the following syntax.

- $\text{GlobalSetup}(1^\lambda, \text{params}) \rightarrow (\text{crs}, \text{msk})$ takes as input the security parameter λ and possibly some parameters params of \mathcal{C} (e.g. the maximal circuit depth), and outputs a common reference string crs and a master secret key msk .

- $\text{Setup}_{\text{crs}}(\text{msk}) \rightarrow (\text{pk}, k_P)$ takes as input a common reference string crs and a master secret key msk , and outputs a public key pk and a proving key k_P .
- $\text{Prove}_{\text{crs}}(C, (\text{pk}, k_P), x, w) \rightarrow \pi$ takes as input a common reference string crs , a circuit $C \in \mathcal{C}$, a public key pk , a proving key k_P , a statement x and a witness w . It outputs a proof π .
- $\text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi) \rightarrow \{0, 1\}$ takes as input a common reference string crs , a circuit $C \in \mathcal{C}$, a public key pk , a statement x and a proof π , and either accepts (with output 1) or rejects (with output 0) the proof.

Moreover, Π_{DPNIZK} should satisfy the following properties:

(Perfect) *Completeness.* For all sufficiently large λ , for all $C \in \mathcal{C}$ and for all pairs (x, w) for which $C(x, w) = 1$, it holds that

$$\begin{aligned} (\text{crs}, \text{msk}) &\leftarrow \text{GlobalSetup}(1^\lambda, \text{params}); \\ (\text{pk}, k_P) &\leftarrow \text{Setup}_{\text{crs}}(\text{msk}); \\ \pi &\leftarrow \text{Prove}_{\text{crs}}(C, (\text{pk}, k_P), x, w); \\ \text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi) &= 1. \end{aligned}$$

(Statistical) *Soundness.* For all sufficiently large λ , for all $C \in \mathcal{C}$ and for all $(\text{crs}, \text{msk}) \leftarrow \text{GlobalSetup}(1^\lambda)$ it holds that

$$\Pr_{\text{pk} \leftarrow \text{Setup}_{\text{crs}}(\text{msk})} \left[\exists (x, \pi) : \begin{array}{c} x \notin \mathcal{L}_C \\ \text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi) = 1 \end{array} \right] = \text{negl}(\lambda).$$

(Statistical) ϵ -*Soundness.* We also define a generalized notion of soundness as follows. For all sufficiently large λ , for all $C \in \mathcal{C}$ for all $(\text{crs}, \text{msk}) \leftarrow \text{GlobalSetup}(1^\lambda)$ it holds that

$$\Pr_{\text{pk} \leftarrow \text{Setup}_{\text{crs}}(\text{msk})} \left[\exists (x, \pi) : \begin{array}{c} x \notin \mathcal{L}_C \\ \text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi) = 1 \end{array} \right] = \epsilon(\lambda).$$

(Programmable CRS) *Single-Statement Zero-Knowledge.* For all PPT adversaries \mathcal{A} there exists a PPT simulator $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ such that

$$\left| \Pr \left[\mathcal{A}^{\{\text{pk}^i\} \leftarrow \text{Setup}_{\text{crs}}(\text{msk})}, \text{Prove}_{\text{crs}}(\cdot, \text{pk}^i, k_P^i, \cdot, \cdot) (\text{crs}) = 1 \right] - \Pr \left[\mathcal{A}^{\{\text{pk}^i\} \leftarrow \mathcal{O}_1}, \mathcal{O}_2(\cdot, \text{pk}^i, \cdot, \cdot) (\text{crs}) = 1 \right] \right| = \text{negl}(\lambda),$$

where

$$\mathcal{O}_1 = (\text{pk}^i, \tau^i) \leftarrow \mathcal{S}_1(\text{crs}); \quad \text{Output } \text{pk}^i;$$

and

$$\mathcal{O}_2(C, \text{pk}^i, x, w) = \begin{cases} \mathcal{S}_2(\text{crs}, C, \text{pk}^i, \tau^i, x) & C \in \mathcal{C} \wedge C(x, w) = 1 \\ \perp & \text{o.w.} \end{cases},$$

the probability is over the coins of $\mathcal{A}, \mathcal{S}, \text{Setup}, \text{Prove}$ and $\text{crs} \leftarrow \text{GlobalSetup}(1^\lambda, \text{params})$, and for every i the adversary \mathcal{A} makes at most a single query of the form $\text{Prove}_{\text{crs}}(\cdot, \text{pk}^i, k_P^i, \cdot, \cdot)$.

Efficiency. For all λ there exists a $p = \text{poly}(\lambda)$ such that for all params and all $(\text{crs}, \text{msk}) \leftarrow \text{GlobalSetup}(1^\lambda, \text{params})$, the complexity of $\text{Setup}_{\text{crs}}(\text{msk})$ is p (and in particular does not depend on params).

Remark 2. A global-setup DP-NIZK can be viewed as a generalization of standard DP-NIZK in the following manner. When the algorithm GlobalSetup is trivial (i.e. when it outputs $\text{crs} = \text{msk} = 1^\lambda$), then the tuple of algorithms $(\text{Setup}, \text{Prove}, \text{Verify})$ qualify as a DP-NIZK proof system with efficient setup and single-statement zero-knowledge.

Remark 3. Every 1DPNIZK with standard statistical soundness can be amplified to 1DPNIZK with statistical ϵ -soundness for any $\epsilon = \frac{1}{2^{\text{poly}(\lambda)}}$ via parallel composition of $\text{Setup}, \text{Prove}, \text{Verify}$ for $\log\left(\frac{1}{\epsilon}\right) = \text{poly}(\lambda)$ times. The single-statement zero-knowledge simulator of the amplified proof system is derived via parallel composition of the simulator of the underlying 1DPNIZK proof system.

4 Our Construction

4.1 Single-Statement Global-Setup DP-NIZK from FHE

Theorem 1. *Assuming the existence of the following building blocks, for every $d \in \mathbb{N}$ there exists a single-statement global-setup DP-NIZK proof system as in Definition 8 for the ensemble \mathcal{C}_d of NP relations that are verifiable by circuits C of depth at most d .*

1. A leveled fully-homomorphic scheme $\text{FHE} = (\text{Keygen}, \text{Enc}, \text{Eval}, \text{Dec})$ with correctness for all ciphertexts as in Definition 5,6. For every λ let $p = \text{poly}(\lambda)$ denote the size of FHE evaluated-ciphertexts and secret-keys as described in the “compactness” section of Definition 5.
2. A garbling scheme $\text{GC} = (\text{Garble}, \text{Eval})$ as in Definition 4 where each label is of size $\lambda = \text{poly}(\lambda)$ bits.
3. A statistically-binding equivocal commitment scheme $\text{SBCS} = (\text{Gen}, \text{Commit}, \text{Ver})$ as in Definition 3.

In the rest of this section we prove Theorem 1. We let $\text{params} = d$ be the depth bound of circuits in \mathcal{C}_d .

Construction 9 (Single-Statement Global-Setup DP-NIZK).

- $\text{GlobalSetup}(1^\lambda, 1^d)$:
 1. Compute $(\text{hpk}, \text{hsk}) \leftarrow \text{FHE.Keygen}(1^\lambda, 1^d)$.
 2. Output $\text{crs} := \text{hpk}$ and $\text{msk} := \text{hsk}$.
- $\text{Setup}_{\text{crs}}(\text{msk})$:
 1. Parse $\text{msk} = \text{hsk}$ and let $D_{\text{hsk}} : \{0, 1\}^p \rightarrow \{0, 1\}$ be the boolean circuit that has hsk hard-wired in it, takes as input an FHE evaluated-ciphertext ct , and decrypts ct with hsk . Compute

$$\left(\widetilde{D}_{\text{hsk}}, \{\text{lab}_{i,b}\}_{i \in [p], b \in \{0,1\}} \right) \leftarrow \text{GC.Garble}(1^\lambda, D_{\text{hsk}}) .$$

2. For $i \in [p]$ and $b \in \{0, 1\}$ compute $\text{crs}_{i,b} \leftarrow \text{SBCS.Setup}(1^\lambda, 1^\lambda)$ and $(c_{i,b}, d_{i,b}) \leftarrow \text{SBCS.Commit}(\text{crs}_{i,b}, \text{lab}_{i,b})$.
3. Output

$$\text{pk} := \left(\widetilde{D}_{\text{hsk}}, \{\text{crs}_{i,b}, c_{i,b}\}_{i \in [p], b \in \{0,1\}} \right), \quad k_P := \{\text{lab}_{i,b}, d_{i,b}\}_{i \in [p], b \in \{0,1\}} .$$

– $\text{Prove}_{\text{crs}}(C, (\text{pk}, k_P), x, w)$:

1. If $C \notin \mathcal{C}_d$ or $C(x, w) \neq 1$ then output \perp .
2. Encrypt $\text{ct}_w \leftarrow \text{FHE.Enc}_{\text{hpk}}(w)$.
3. Let C'_x be the circuit $C_x(\circ) := C(x, \circ)$ and compute homomorphically

$$\text{ct}_b \leftarrow \text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, C_x) .$$

4. Note that $\text{ct}_b \in \{0, 1\}^p$. For $i \in [p]$ let \tilde{b}_i denote the i th bit of ct_b . Output

$$\pi := \left(\text{ct}_w, \{\text{lab}_{i, \tilde{b}_i}, d_{i, \tilde{b}_i}\}_{i \in [p]} \right) .$$

– $\text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi)$:

1. Parse $\text{crs} = \text{hpk}$, $\text{pk} = \left(\widetilde{D}_{\text{hsk}}, \{\text{crs}_{i,b}, c_{i,b}\}_{i \in [p], b \in \{0,1\}} \right)$ and $\pi = (\text{ct}_w, \{\text{lab}_i, d_i\}_{i \in [p]})$.
2. Compute $\text{ct}_b \leftarrow \text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, C_x)$ (where C_x is as defined above).
3. Note that $\text{ct}_b \in \{0, 1\}^p$. For $i \in [p]$ let \tilde{b}_i denote the i th bit of ct_b . Verify the label decommitments: for $i \in [p]$ compute

$$\text{SBCS.Ver} \left(\text{crs}_{i, \tilde{b}_i}, c_{i, \tilde{b}_i}, \text{lab}_i, d_i \right) ,$$

if any of those verifications fail then output 0 (reject).

4. Compute and output $\text{GC.Eval} \left(\widetilde{D}_{\text{sk}}, \{\text{lab}_i\}_{i \in [p]} \right)$.

Proof of Completeness. Fix λ, d, C, x, w where $C \in \mathcal{C}_d$ and $C(x, w) = 1$. Consider

$$(\text{crs}, \text{msk}) \leftarrow \text{GlobalSetup}(1^\lambda, 1^d) ,$$

$$(\text{pk}, k_P) \leftarrow \text{Setup}_{\text{crs}}(\text{msk}) ,$$

$$\pi \leftarrow \text{Prove}_{\text{crs}}(C, (\text{pk}, k_P), x, w) .$$

Parse $\text{crs} = \text{hpk}$, $\text{pk} = \left(\widetilde{D}_{\text{hsk}}, \{\text{crs}_{i,b}, c_{i,b}\}_{i \in [p], b \in \{0,1\}} \right)$ and $\pi = (\text{ct}_w, \{\text{lab}_i, d_i\}_{i \in [p]})$.

Consider the execution of $\text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi)$. Since FHE.Eval is deterministic, the value ct_b that is computed in Prove and in Verify is identical. Therefore the correctness of SBCS implies that all of the decommitment verifications in step (3) of Verify pass. Moreover, due to the correctness of FHE it holds that

$$\begin{aligned} \text{FHE.Dec}_{\text{hsk}}(\text{ct}_b) &= \text{FHE.Dec}_{\text{hsk}}(\text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, C_x)) \\ &= \text{FHE.Dec}_{\text{hsk}}(\text{FHE.Eval}_{\text{hpk}}(\text{FHE.Enc}_{\text{hpk}}(w), C_x)) \\ &= C_x(w) = C(x, w) = 1 , \end{aligned}$$

and the correctness of GC implies that the output in step (4) of Verify is $\text{FHE.Dec}_{\text{hsk}}(\text{ct}_b) = 1$. \square

Proof of Soundness. Fix $\lambda, d \in \mathbb{N}$, $C \in \mathcal{C}_d$ and $(\text{crs}, \text{msk}) \leftarrow \text{GlobalSetup}(1^\lambda, 1^d)$, and consider the random variable $\text{pk} \leftarrow \text{Setup}_{\text{crs}}(\text{msk})$. Assume that there exist (x, π) such that $x \notin \mathcal{L}_C$ and $\text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi) = 1$.

Parse $\text{crs} = \text{hpk}$, $\text{pk} = (\widetilde{D}_{\text{hsk}}, \{\text{crs}_{i,b}, c_{i,b}\}_{i \in [p], b \in \{0,1\}})$ and $\pi = (\text{ct}_w, \{\text{lab}'_i, d'_i\}_{i \in [p]})$, and recall that the values in the pk were computed as follows

$$(\widetilde{D}_{\text{hsk}}, \{\text{lab}_{i,b}\}_{i \in [p], b \in \{0,1\}}) \leftarrow \text{GC.Garble}(1^\lambda, D_{\text{hsk}})$$

and

$$\forall i \in [p] : (c_{i,b}, d_{i,b}) \leftarrow \text{SBCS.Commit}(\text{crs}_{i,b}, \text{lab}_{i,b}) .$$

Let $\text{ct}_b \leftarrow \text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, C_x)$ be the value that is computed during `Verify` and for $i \in [p]$ let \widetilde{b}_i denote the i th bit of ct_b .

Assume towards contradiction that for all $i \in [p]$ it holds that $\text{lab}'_i = \text{lab}_{i, \widetilde{b}_i}$, then by the correctness of GC and FHE, and since `Verify` outputs 1, it holds that

$$\begin{aligned} 1 &= \text{GC.Eval}(\widetilde{D}_{\text{hsk}}, \{\text{lab}'_i\}_{i \in [p]}) \\ &= \text{GC.Eval}(\widetilde{D}_{\text{hsk}}, \{\text{lab}_{i, \widetilde{b}_i}\}_{i \in [p]}) \\ &= \text{FHE.Dec}_{\text{hsk}}(\text{ct}_b) \\ &= \text{FHE.Dec}_{\text{hsk}}(\text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, C_x)) \\ &= C_x(\text{FHE.Dec}_{\text{hsk}}(\text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, I))) \\ &= C(x, \text{FHE.Dec}_{\text{hsk}}(\text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, I))) , \end{aligned}$$

and therefore the string $w := \text{FHE.Dec}_{\text{hsk}}(\text{FHE.Eval}_{\text{hpk}}(\text{ct}_w, I))$ satisfies $C(x, w) = 1$, with contradiction to the assumption that $x \notin \mathcal{L}_C$.

Therefore, it must be the case that there exists some $j \in [p]$ for which $\text{lab}'_j \neq \text{lab}_{j, \widetilde{b}_j}$. Since all of the verifications in step (3) of `Verify` pass successfully, it in particular holds that

$$\text{SBCS.Ver}(c_{j, \widetilde{b}_j}, c_{j, \widetilde{b}_j}, \text{lab}'_j, d'_j) = 1 .$$

Therefore, denoting $\text{crs}^* := \text{crs}_{j, \widetilde{b}_j}$, $m_0^* := \text{lab}_{j, \widetilde{b}_j}$, $m_1^* := \text{lab}'_j$ and $d^* := d'_j$, and letting r^* be the randomness used during $\text{pk} \leftarrow \text{Setup}_{\text{crs}}(\text{msk})$ when computing $c_{j, \widetilde{b}_j} \leftarrow \text{SBCS.Commit}(\text{crs}_{j, \widetilde{b}_j}, \text{lab}_{j, \widetilde{b}_j}; r^*)$, it holds that

$$\begin{aligned} Pr_{\text{pk} \leftarrow \text{Setup}_{\text{crs}}(\text{msk})} \left[\exists (x, \pi) : \begin{array}{l} x \notin \mathcal{L}_C \\ \text{Verify}_{\text{crs}}(C, \text{pk}, x, \pi) = 1 \end{array} \right] &\leq \\ Pr_{\text{crs}^* \leftarrow \text{SBCS.Setup}(1^\lambda, 1^\lambda)} \left[\exists (r^*, m_0^*, m_1^*, d^*) : \begin{array}{l} c^* \leftarrow \text{SBCS.Commit}(\text{crs}^*, m_0^*; r^*) \\ m_0^* \neq m_1^* \\ \text{SBCS.Verify}(\text{crs}^*, c^*, m_1^*, d^*) = 1 \end{array} \right] &= \text{negl}(\lambda) \end{aligned}$$

where the last equation is due to the binding of SBCS. \square

Proof of Single-Statement Zero-Knowledge Let $\text{SBCS}.\mathcal{S} = (\text{SBCS}.\mathcal{S}^A, \text{SBCS}.\mathcal{S}^B)$ be the equivocability simulator of SBCS and let $\text{GC}.\mathcal{S}$ be the simulator of the garbling scheme. Define the single-statement zero-knowledge simulator $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ as follows:

– $\mathcal{S}_1(\text{crs})$:

1. Set $\text{hsk} := 0^p$, where p is the upper-bound on the size of FHE secret-keys and evaluated ciphertexts, as in Definition 5.
2. Let $D_{\text{hsk}} : \{0, 1\}^p \rightarrow \{0, 1\}$ be the boolean circuit that has hsk hard-wired in it, takes as input an FHE evaluated-ciphertext ct , and decrypts ct with hsk . Compute

$$\left(\widetilde{D}_{\text{hsk}}, \{\text{lab}_i\}_{i \in [p]}\right) \leftarrow \mathcal{S}_{\text{GC}}\left(1^\lambda, 1^{|\text{D}_{\text{hsk}}|}, 1\right).$$

3. For $i \in [p]$ and $b \in \{0, 1\}$ compute $(\text{crs}_{i,b}, c_{i,b}, \text{td}_{i,b}^c) \leftarrow \text{SBCS}.\mathcal{S}^A(1^\lambda, 1^\lambda)$.
4. Set

$$\text{pk} := \left(\widetilde{D}_{\text{hsk}}, \{c_{i,b}\}_{i \in [p], b \in \{0,1\}}\right), \quad \tau := \left(\{\text{lab}_i\}_{i \in [p]}, \{\text{td}_{i,b}^c\}_{i \in [p], b \in \{0,1\}}\right).$$

– $\mathcal{S}_2(\text{crs}, C, \text{pk}, \tau, x)$:

1. Parse $\text{crs} = \text{hpk}$, $\text{pk} = \left(\widetilde{D}_{\text{hsk}}, \{c_{i,b}\}_{i \in [p], b \in \{0,1\}}\right)$ and $\tau = \left(\{\text{lab}_i\}_{i \in [p]}, \{\text{td}_{i,b}^c\}_{i \in [p], b \in \{0,1\}}\right)$.
2. Encrypt $\text{ct}_w \leftarrow \text{FHE}.\text{Enc}_{\text{hpk}}(0^k)$, where k is the bit-length of witnesses as determined by C .
3. Compute homomorphically $\text{ct}_b \leftarrow \text{FHE}.\text{Eval}_{\text{hpk}}(\text{ct}_w, C_x)$, where C_x is the circuit $C_x(\circ) := C(x, \circ)$.
4. Note that $\text{ct}_b \in \{0, 1\}^p$. For $i \in [p]$ let \tilde{b}_i denote the i th bit of ct_b and compute

$$d_i \leftarrow \text{SBCS}.\mathcal{S}^B\left(\text{crs}_{i,\tilde{b}_i}, c_{i,\tilde{b}_i}, \text{td}_{i,\tilde{b}_i}^c, \text{lab}_i\right)$$

5. Output

$$\pi := (\text{ct}_w, \{\text{lab}_i, d_i\}_{i \in [p]}).$$

We now prove indistinguishability via a sequence of hybrids:

Hybrid \mathcal{H}_0 . The real **Setup**, **Prove** algorithms.

Hybrid \mathcal{H}_1 . We change the way that the values $\{\text{crs}_{i,b}, c_{i,b}\}_{i \in [p], b \in \{0,1\}}$ and $\{d_{i,\tilde{b}_i}\}_{i \in [p]}$ are computed in **Setup** and **Prove** respectively:

1. In **Setup**, for $i \in [p]$ and $b \in \{0, 1\}$ compute $(\text{crs}_{i,b}, c_{i,b}, \text{td}_{i,b}^c) \leftarrow \text{SBCS}.\mathcal{S}^A(1^\lambda, 1^\lambda)$.
2. In **Prove**, for $i \in [p]$ compute

$$d_i \leftarrow \text{SBCS}.\mathcal{S}^B\left(\text{crs}_{i,\tilde{b}_i}, c_{i,\tilde{b}_i}, \text{td}_{i,\tilde{b}_i}^c, \text{lab}_{i,\tilde{b}_i}\right).$$

Hybrids \mathcal{H}_1 and \mathcal{H}_0 are computationally indistinguishable due to the equivocability of SBCS.

Hybrid \mathcal{H}_2 . Note that in Hybrid \mathcal{H}_1 the values $\{\text{lab}_{i,\tilde{b}_i}\}_{i \in [p]}$ are only used during Prove, and the other p GC labels are never used. In this hybrid we change the way that the values $\widetilde{D}_{\text{hsk}}$ and $\{\text{lab}_{i,\tilde{b}_i}\}_{i \in [p]}$ are computed in Setup and Prove respectively:

1. In Setup, compute

$$\left(\widetilde{D}_{\text{hsk}}, \{\text{lab}_i\}_{i \in [p]}\right) \leftarrow \mathcal{S}_{\text{GC}}\left(1^\lambda, 1^{|\widetilde{D}_{\text{hsk}}|}, 1\right).$$

2. In Prove, for $i \in [p]$ set $\text{lab}_{i,\tilde{b}_i} := \text{lab}_i$ and proceed as in the previous hybrid.

Hybrids \mathcal{H}_2 and \mathcal{H}_1 are computationally indistinguishable due to the security of GC.

Hybrid \mathcal{H}_3 . Note that in Hybrid \mathcal{H}_2 the value hsk is only used when computing the FHE ciphertext ct_w in Prove. In this hybrids we change the way that ct_w is computed: Encrypt $\text{ct}_w \leftarrow \text{FHE.Enc}_{\text{hpk}}(0^k)$, where k is the bit-length of witnesses as determined by C . Hybrids \mathcal{H}_3 and \mathcal{H}_2 are computationally indistinguishable due to the security of FHE.

Note that this hybrid is identical to the simulators $\mathcal{S}_1, \mathcal{S}_2$. \square

Efficiency. Fix λ and note that by the compactness of FHE, there exists some $p = \text{poly}(\lambda)$ such that for all $\text{params} = d$ and $(\text{hpk}, \text{hsk}) \leftarrow \text{FHE.Keygen}(1^\lambda, 1^d)$, the size of FHE evaluated-ciphertexts and hsk is at most p . Denote $(\text{crs}, \text{msk}) = (\text{hpk}, \text{hsk})$ and note that the running time of $\text{Setup}_{\text{crs}}(\text{msk})$ is bounded by some $p' = \text{poly}(p, \lambda) = \text{poly}(\lambda)$, i.e. for all $\text{params} = d$ and all $(\text{crs}, \text{msk}) \leftarrow \text{GlobalSetup}(1^\lambda, 1^d)$, the complexity of $\text{Setup}_{\text{crs}}(\text{msk})$ is at most p' .

4.2 DP-NIZK from Single-Statement Global-Setup DP-NIZK

Theorem 2. *Assuming the existence of the following building blocks, for every $d \in \mathbb{N}$ there exists a DPNIZK proof system as in Definition 7 for the ensemble \mathcal{C}_d of NP relations that are verifiable by circuits C of depth at most d .*

1. A pseudo-random function $\text{PRF} = (\text{Setup}, \text{Eval})$ where w.l.o.g. for every $k \leftarrow \text{PRF.Setup}(1^\lambda)$ it holds that $k \in \{0, 1\}^\lambda$.
2. A single-statement global-setup DPNIZK proof system $1\text{DPNIZK} = (\text{GlobalSetup}, \text{Setup}, \text{Prove}, \text{Verify})$ for $\{\mathcal{C}_d\}_d$, where w.l.o.g. for every $(\text{crs}, \text{msk}) \leftarrow 1\text{DPNIZK.GlobalSetup}(1^\lambda, 1^d)$ it holds that the randomness used by $1\text{DPNIZK.Setup}_{\text{crs}}(\text{msk})$ is of size $\ell = \text{poly}(\lambda)$, the size of msk is some $p = \text{poly}(\lambda)$ and the scheme satisfies $(2^{\lambda-\ell}, \lambda)$ -soundness.
3. A statistically-binding commitment scheme $\text{SBCS} = (\text{Gen}, \text{Commit}, \text{Ver})$ as in Definition 3.

In the rest of this section we prove Theorem 2.

Construction 10 (DP-NIZK from 1-DP-NIZK).

- **Setup**($1^\lambda, 1^d$):
1. For $i = 0, \dots, \lambda$ compute $(\text{crs}'_i, \text{msk}'_i) \leftarrow \text{1DPNIZK.GlobalSetup}(1^\lambda, 1^{d'_i})$ where d'_i is defined in the paragraph below.
 2. Sample $r \xleftarrow{\$} \{0, 1\}^\ell$ and for $i \in [\lambda]$ sample $k_i \leftarrow \text{PRF.Setup}(1^\lambda)$.
 3. Compute $\text{crs}^* \leftarrow \text{SBCS.Gen}(1^\lambda, 1^{p+\lambda})$ and for $i \in [\lambda]$ sample $(c_i^*, d_i^*) \leftarrow \text{SBCS.Commit}(\text{crs}^*, (\text{msk}'_i, k_i))$.
 4. Sample $(\text{pk}^\emptyset, k_P^\emptyset) \leftarrow \text{1DPNIZK.Setup}_{\text{crs}'_0}(\text{msk}'_0)$.
 5. Output $\text{crs} := (\{\text{crs}'_i\}_{i=0, \dots, \lambda}, \text{crs}^*, \{c_i^*\}_{i \in [\lambda]}, \text{pk}^\emptyset, r)$ and $k_P := (\{\text{msk}'_i\}_{i=0, \dots, \lambda}, \{k_i, d_i^*\}_{i \in [\lambda]}, k_P^\emptyset)$.
- **Prove** $_{\text{crs}}(C, k_P, x, w)$:
1. Parse $\text{crs} = (\{\text{crs}'_i\}_{i=0, \dots, \lambda}, \text{crs}^*, \{c_i^*\}_{i \in [\lambda]}, \text{pk}^\emptyset, r)$ and $k_P = (\{\text{msk}'_i\}_{i=0, \dots, \lambda}, \{k_i, d_i^*\}_{i \in [\lambda]}, k_P^\emptyset)$.
 2. Sample $m \xleftarrow{\$} \{0, 1\}^\lambda$ and for $i \in [\lambda]$ let m^i denote length- i prefix of m (i.e. $m^i = m_1 m_2 \dots m_i$). In particular denote $m^0 = \emptyset$ and $m^\lambda = m$.
 3. For $i = 0, \dots, \lambda - 1$ do:
 - (a) For $b \in \{0, 1\}$ compute

$$r^{m^i \| b} := r \oplus \text{PRF.Eval}_{k_{i+1}}(m^i \| b)$$
 and sample a 1-DPNIZK instance respective to $(\text{crs}'_{i+1}, \text{msk}'_{i+1})$ with $r^{m^i \| b}$ as randomness:

$$(\text{pk}^{m^i \| b}, k_P^{m^i \| b}) := \text{1DPNIZK.Setup}_{\text{crs}'_{i+1}}(\text{msk}'_{i+1} ; r^{m^i \| b}) .$$
 - (b) Let C'_i be the relation that takes as a statement a pair (\circ_0, \circ_1) and as a witness a 3-tuple $(\bullet_0, \bullet_1, \bullet_2)$, and outputs 1 iff

$$\text{SBCS.Ver}(\text{crs}^*, c_{i+1}^*, (\bullet_0, \bullet_1), \bullet_2) = \text{accept} \quad \wedge$$

$$\forall b \in \{0, 1\}, \quad \circ_b = \text{1DPNIZK.Setup}_{\text{crs}'_{i+1}}(\bullet_0 ; r \oplus \text{PRF.Eval}_{\bullet_1}(m^i \| b)) .$$
 Compute

$$\pi^{m^i} \leftarrow \text{1DPNIZK.Prove}_{\text{crs}'_i}(C'_i, (\text{pk}^{m^i}, k_P^{m^i}), (\text{pk}^{m^i \| 0}, \text{pk}^{m^i \| 1}), (\text{msk}'_{i+1}, k_{i+1}, d_{i+1}^*)) .$$
 4. Compute

$$\pi^m \leftarrow \text{1DPNIZK.Prove}_{\text{crs}'_\lambda}(C, (\text{pk}^m, k_P^m), x, w) .$$
 5. Output $\pi := (m, \{\text{pk}^{m^i \| b}\}_{i=0, \dots, \lambda-1, b \in \{0, 1\}}, \{\pi^{m^i}\}_{i=0, \dots, \lambda})$.
- **Verify** $_{\text{crs}}(C, x, \pi)$:
1. Parse $\text{crs} = (\{\text{crs}'_i\}_{i=0, \dots, \lambda}, \text{crs}^*, \{c_i^*\}_{i \in [\lambda]}, \text{pk}^\emptyset, r)$ and $\pi = (m, \{\text{pk}^{m^i \| b}\}_{i=0, \dots, \lambda-1, b \in \{0, 1\}}, \{\pi^{m^i}\}_{i=0, \dots, \lambda})$.
 2. For $i = 0, \dots, \lambda - 1$ compute

$$\text{1DPNIZK.Verify}_{\text{crs}'_i}(C'_i, \text{pk}^{m^i}, (\text{pk}^{m^i \| 0}, \text{pk}^{m^i \| 1}), \pi^{m^i})$$
 and if it rejects (outputs 0) then reject (output 0).
 3. Compute and output

$$\text{1DPNIZK.Verify}_{\text{crs}'_\lambda}(C, \text{pk}^m, x, \pi^m) .$$

Choice of Parameters. Note that by the efficiency of 1DPNIZK there is some fixed polynomial $p = p(\lambda)$ such that for all λ, d and $(\text{crs}', \text{msk}') \leftarrow \text{1DPNIZK.GlobalSetup}(1^\lambda, 1^d)$ the complexity of $\text{1DPNIZK.Setup}_{\text{crs}'}(\text{msk}')$ is p . Therefore, there is some fixed polynomial $p' = \text{poly}(\lambda, p) = \text{poly}(\lambda)$ such that for all λ, d and $(\text{crs}, k_P) \leftarrow \text{Setup}(1^\lambda, 1^d)$, the complexity of $\{C'_i\}_i$ (the circuits defined in step (b) of $\text{Prove}_{\text{crs}}(\cdot, k_P, \cdot, \cdot)$) is at most p' . It follows that there is also some $d'' = \text{poly}(\lambda)$ such that for all λ, d and $(\text{crs}, k_P) \leftarrow \text{Setup}(1^\lambda, 1^d)$, the *depth* of $\{C'_i\}_i$ is at most d'' . For $i < \lambda$ we set $d'_i := d''$ and for $i = \lambda$ we set $d'_\lambda := d$.

Proof of Completeness. Fix λ, d, C, x, w where C is of depth at most d and $C(x, w) = 1$. Consider $(\text{crs}, k_P) \leftarrow \text{Setup}(1^\lambda, 1^d)$ and $\pi \leftarrow \text{Prove}_{\text{crs}}(C, k_P, x, w)$. Parse

$$\text{crs} = (\{\text{crs}'_i\}_{i=0, \dots, \lambda}, \text{crs}^*, \{c_i^*\}_{i \in [\lambda]}, \text{pk}^\emptyset, r), \quad \pi = \left(\{\text{pk}^{m^i \| b}\}_{i=0, \dots, \lambda-1, b \in \{0,1\}}, \{\pi^{m^i}\}_{i=0, \dots, \lambda} \right).$$

Consider the execution of $\text{Verify}_{\text{crs}}(C, x, \pi)$. For $i = 0, \dots, \lambda - 1$ it holds that

$$C'_i \left((\text{pk}^{m^i \| 0}, \text{pk}^{m^i \| 1}), (\text{msk}'_{i+1}, k_{i+1}, d_{i+1}^*) \right) = 1$$

and therefore $\text{1DPNIZK.Verify}_{\text{crs}'_i}(C'_i, (\text{pk}^{m^i \| 0}, \text{pk}^{m^i \| 1}), \pi^{m^i}) = 1$.

Moreover, since $C(x, w) = 1$, it holds that $\text{1DPNIZK.Verify}_{\text{crs}'_\lambda}(C, x, \pi^m) = 1$. \square

Proof of (Statistical) Soundness.

Notation. For any fixed pair $(\text{crs}', \text{msk}') \leftarrow \text{1DPNIZK.Global.Setup}(1^\lambda, \text{params})$ we divide the space $\{0, 1\}^\ell$ into “good randomness” and “bad randomness”, where a string $s' \in \{0, 1\}^\ell$ is “bad randomness” respective to $(\text{crs}', \text{msk}')$ if it breaks its soundness, i.e. if

$$\exists(C, x, \pi) : x \notin \mathcal{L}_C \wedge \text{1DPNIZK.Verify}_{\text{crs}'}(C, \text{pk}', x, \pi) = 1$$

where $\text{pk}' \leftarrow \text{1DPNIZK.Setup}_{\text{crs}'}(\text{msk}' ; s')$, and otherwise s' is “good randomness”.

The following lemma follows immediately from the ϵ -soundness of 1DPNIZK if $\epsilon(\lambda) = 2^{-\lambda} \cdot \text{negl}(\lambda)$:

Lemma 2. *For every pair $(\text{crs}', \text{msk}') \leftarrow \text{1DPNIZK.Global.Setup}(1^\lambda, 1^d)$ and every set $S \subset \{0, 1\}^\ell$ of size at most 2^λ ,*

$$\Pr_{r \xleftarrow{\$} \{0,1\}^\ell} [\exists s \in S, s \oplus r \text{ is bad randomness respective to } (\text{crs}', \text{msk}')] = \text{negl}(\lambda).$$

We now proceed with the proof of soundness. Fix λ, d and a circuit $C \in \mathcal{C}_d$. Consider the random variable $\text{crs} \leftarrow \text{Setup}(1^\lambda, 1^d)$ and the corresponding circuits $\{C'_i\}_{i=0, \dots, \lambda-1}$ as described in step (b) of $\text{Prove}_{\text{crs}}$. Parse

$$\text{crs} = (\{\text{crs}'_i\}_{i=0, \dots, \lambda}, \text{crs}^*, \{c_i^*\}_{i \in [\lambda]}, \text{pk}^\emptyset, r)$$

and recall that crs'_i was computed as $(\text{crs}'_i, \text{msk}'_i) \leftarrow \text{1DPNIZK.GlobalSetup}(1^\lambda, 1^{d'_i})$ and $r \xleftarrow{\$} \{0, 1\}^\ell$. Moreover, the values $\{c_i^*\}_{i \in [\lambda]}$ were computed as

$$(c_i^*, d_i^*) \leftarrow \text{SBCS.Commit}(\text{crs}^*, (\text{msk}'_i, k_i))$$

where $\text{crs}^* \leftarrow \text{SBCS.Gen}(1^\lambda, 1^{p+\lambda})$ and $k_i \leftarrow \text{PRF.Setup}(1^\lambda)$.

For all $i \in [\lambda]$ consider the set of strings $S^i := \{\text{PRF.Eval}_{k_i}(m^i)\}_{m^i \in \{0,1\}^i}$. Then due to Lemma 2, it holds that

$$\Pr_{r \xleftarrow{\$} \{0,1\}^\ell} [\exists s \in S^i, s \oplus r \text{ is bad randomness relative to } (\text{crs}'_i, \text{msk}'_i)] = \text{negl}(\lambda) . \quad (1)$$

Assume that there exist (x, π) such that $x \notin \mathcal{L}_C$ and $\text{Verify}_{\text{crs}}(C, x, \pi) = 1$. Parse

$$\pi = \left(m, \{\text{pk}^{m^i \| b}\}_{i=0, \dots, \lambda-1, b \in \{0,1\}}, \{\pi^{m^i}\}_{i=0, \dots, \lambda} \right) .$$

- Assume that there exists some $j \in [\lambda]$ such that the value $\text{pk}^{m^{j-1}}$ (as appears in π) was computed “honestly” and the value pk^{m^j} (as appears in π) wasn’t computed “honestly”, i.e. assume that

$$\text{pk}^{m^{j-1}} = \begin{cases} \text{1DPNIZK.Setup}_{\text{crs}'_{j-1}}(\text{msk}'_{j-1} ; r \oplus \text{PRF.Eval}_{k_{j-1}}(m^{j-1})) & j-1 > 0 \\ \text{1DPNIZK.Setup}_{\text{crs}'_0}(\text{msk}'_0 ; s \xleftarrow{\$} \{0, 1\}^\ell) & j-1 = 0 \end{cases}$$

and

$$\text{pk}^{m^j} \neq \text{1DPNIZK.Setup}_{\text{crs}'_j}(\text{msk}'_j ; r \oplus \text{PRF.Eval}_{k_j}(m^j)) .$$

Due to soundness of 1DPNIZK relative to $(\text{crs}'_{j-1}, \text{msk}'_{j-1})$ and $\text{pk}^{m^{j-1}}$ (which holds with all but negl. prob due to Eq. (1)), and since we assume that $\text{Verify}_{\text{crs}}(C, x, \pi) = 1$ and in particular that $\text{1DPNIZK.Verify}_{\text{crs}'_{j-1}}(C'_{j-1}, \text{pk}^{m^{j-1}}, (\text{pk}^{m^{j-1} \| 0}, \text{pk}^{m^{j-1} \| 1}), \pi^{m^{j-1}}) = 1$, with all but negl. prob, there exists a string \hat{w}_{j-1} such that

$$C'_{j-1} \left((\text{pk}^{m^{j-1} \| 0}, \text{pk}^{m^{j-1} \| 1}), \hat{w}_{j-1} \right) = 1 . \quad (2)$$

Parse $\hat{w}_{j-1} = (\widehat{\text{msk}}_j, \hat{k}_j, \hat{d}_j^*)$, then Eq. (2) in particular means that

$$\text{pk}^{m^j} = \text{1DPNIZK.Setup}_{\text{crs}'_j}(\widehat{\text{msk}}_j ; r \oplus \text{PRF.Eval}_{\hat{k}_j}(m^j)) .$$

Since we assume that pk^{m^j} wasn’t generated honestly, i.e. that

$$\text{pk}^{m^j} \neq \text{1DPNIZK.Setup}_{\text{crs}'_j}(\text{msk}'_j ; r \oplus \text{PRF.Eval}_{k_j}(m^j)) ,$$

it follows that $(\widehat{\text{msk}}_j, \hat{k}_j) \neq (\text{msk}'_j, k_j)$. However, Eq. (2) also implies that

$$\text{SBCS.Ver} \left(\text{crs}^*, c_j^*, (\widehat{\text{msk}}_j, \hat{k}_j), \hat{d}_j^* \right) = \text{accept} ,$$

and therefore the decommitment \hat{d}_j^* breaks the soundness of SBCS relative to (crs^*, c_j^*) and the pair of messages (msk'_j, k_j) and $(\widehat{\text{msk}}_j, \hat{k}_j)$. Since the soundness of SBCS relative to (crs^*, c_j^*) holds with all but negligible probability, it follows that the probability that $\text{pk}^{m^{j-1}}$ was computed “honestly” and pk^{m^j} wasn’t computed “honestly” is negligible.

Since for $j - 1 = 0$ the value $\text{pk}^{m^{j-1}}$ is always generated honestly during Setup, an inductive argument implies that with all but negligible probability all of the values $\text{pk}^\emptyset, \text{pk}^{m^1}, \dots, \text{pk}^{m^{\lambda-1}}, \text{pk}^m \in \pi$ were generated honestly.

Lastly, the soundness of 1DPNIZK relative to $(\text{crs}'_\lambda, \text{msk}'_\lambda)$ and pk^m implies that with all but negligible probability there is no (x, C, π^m) such that

$$x \notin \mathcal{L}_C \wedge \text{1DPNIZK.Verify}_{\text{crs}'_\lambda}(C, \text{pk}^m, x, \pi^m) = 1 .$$

□

Proof of (Programmable CRS) Zero-Knowledge. Let $\text{1DPNIZK.S} = (\mathcal{S}_1, \mathcal{S}_2)$ be the single-statement zero-knowledge simulator of 1DPNIZK and define the zero-knowledge simulator $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ as follows:

- $\mathcal{S}_1(1^\lambda)$:
 1. For $i = 0, \dots, \lambda$ compute $(\text{crs}'_i, \text{msk}'_i) \leftarrow \text{1DPNIZK.GlobalSetup}(1^\lambda, 1^{d'_i})$.
 2. Sample $r \xleftarrow{\$} \{0, 1\}^\ell$.
 3. Compute $\text{crs}^* \leftarrow \text{SBCS.Gen}(1^\lambda, 1^{p+\lambda})$ and for $i \in [\lambda]$ sample $(c_i^*, d_i^*) \leftarrow \text{SBCS.Commit}(\text{crs}^*, 0^{p+\lambda})$.
 4. Compute $(\text{pk}^\emptyset, \tau^\emptyset) \leftarrow \text{1DPNIZK.S}_1(\text{crs}'_0)$.
 5. Output $\text{crs} := (\{\text{crs}'_i\}_{i=0, \dots, \lambda}, \text{crs}^*, \{c_i^*\}_{i \in [\lambda]}, \text{pk}^\emptyset, r)$ and $\tau := \tau^\emptyset$.
- $\mathcal{S}_2(C, \text{crs}, \tau, x)$:
 1. Parse $\text{crs} = (\{\text{crs}'_i\}_{i=0, \dots, \lambda}, \text{crs}^*, \{c_i^*\}_{i \in [\lambda]}, \text{pk}^\emptyset, r)$ and $\tau = \tau^\emptyset$.
 2. Compute $m \xleftarrow{\$} \{0, 1\}^\lambda$ and for $i \in [\lambda]$ let m^i denote length- i prefix of m (i.e. $m^i = m_1 m_2 \dots m_i$). In particular denote $m^0 = \emptyset$ and $m^\lambda = m$.
 3. For $i = 0, \dots, \lambda - 1$ do:
 - (a) For $b \in \{0, 1\}$ compute

$$(\text{pk}^{m^i \| b}, \tau^{m^i \| b}) \leftarrow \text{1DPNIZK.S}_1(\text{crs}'_{i+1}) .$$
 - (b) Compute

$$\pi^{m^i} \leftarrow \text{1DPNIZK.S}_2(\text{crs}'_i, C'_i, \text{pk}^{m^i}, \tau^{m^i}, (\text{pk}^{m^i \| 0}, \text{pk}^{m^i \| 1})) .$$
 4. Compute

$$\pi^m \leftarrow \text{1DPNIZK.S}_2(\text{crs}'_\lambda, C, \text{pk}^m, \tau^m, x) .$$
 5. Output $\pi := (m, \{\text{pk}^{m^i \| b}\}_{i=0, \dots, \lambda-1, b \in \{0, 1\}}, \{\pi^{m^i}\}_{i=0, \dots, \lambda})$.

We now prove indistinguishability via a sequence of $2 + 3\lambda$ hybrids:

Hybrid \mathcal{H}_0 . The real Setup, Prove algorithms.

For $i = 0, \dots, \lambda - 1$ we define the hybrids $\{\mathcal{H}_{i,j}\}_{j \in [3]}$ and consider the sequence

$$\mathcal{H}_0, (\mathcal{H}_{0,1}, \mathcal{H}_{0,2}, \mathcal{H}_{0,3}), (\mathcal{H}_{1,1}, \mathcal{H}_{1,2}, \mathcal{H}_{1,3}), \dots, (\mathcal{H}_{\lambda-1,1}, \mathcal{H}_{\lambda-1,2}, \mathcal{H}_{\lambda-1,3}), \mathcal{H}_\lambda .$$

Hybrid $\mathcal{H}_{i,1}$. Note that in the previous hybrid, 1DPNIZK public-keys respective to crs'_i are sampled with real randomness, and msk'_i is not used elsewhere. We therefore can simulate them and proofs respective to them. Formally, we change the way that values of the form $\text{pk}^{m^{i-1}\|b}$ and π^{m^i} are generated:

- If $i = 0$, change the way that pk^\emptyset is generated during Setup:

$$(\text{pk}^\emptyset, \tau^\emptyset) \leftarrow \text{1DPNIZK.S}_1(\text{crs}'_0) .$$

If $i > 0$, change the way that $\text{pk}^{m^{i-1}\|b}$ is generated during the $(i - 1)$ th iteration of Step (3) of Prove:

$$(\text{pk}^{m^{i-1}\|b}, \tau^{m^{i-1}\|b}) \leftarrow \text{1DPNIZK.S}_1(\text{crs}'_i) .$$

- Change the way that π^{m^i} is generated during the i th iteration of Step (3) of Prove:

$$\pi^{m^i} \leftarrow \text{1DPNIZK.S}_2 \left(\text{crs}'_i, C'_i, \text{pk}^{m^i}, \tau^{m^i}, (pk^{m^i\|0}, \text{pk}^{m^i\|1}) \right) .$$

Hybrids $\mathcal{H}_{i-1,3}$ and $\mathcal{H}_{i,1}$ are indistinguishable due to the single-statement zero-knowledge of 1DPNIZK respective to crs'_i .

Hybrid $\mathcal{H}_{i,2}$. Note that in the previous hybrid, the value d_{i+1}^* is never used. In this hybrid we change the way that the commitment c_{i+1}^* is computed:

$$(c_{i+1}^*, d_{i+1}^*) \leftarrow \text{SBCS.Commit}(\text{crs}^*, 0^{p+\lambda}) .$$

Hybrids $\mathcal{H}_{i,1}$ and $\mathcal{H}_{i,2}$ are indistinguishable due to the hiding of SBCS.

Hybrid $\mathcal{H}_{i,3}$. Note that in the previous hybrid, the value k_{i+1} is only used when computing

$$r^{m^i\|b} := r \oplus \text{PRF.Eval}_{k_{i+1}}(m^i\|b)$$

during Prove. In this hybrid we sample instead $r^{m^i\|b} \xleftarrow{\$} \{0, 1\}^\ell$. Hybrids $\mathcal{H}_{i,2}$ and $\mathcal{H}_{i,3}$ are indistinguishable due to the pseudorandomness of PRF.

Hybrid \mathcal{H}_λ . Note that in the previous hybrid ($\mathcal{H}_{\lambda-1,3}$), 1DPNIZK public-keys respective to crs'_λ are sampled with real randomness, and msk'_λ is not used elsewhere.

Moreover, the values m which are used by the prover when answering proof queries are sampled uniformly at random from $\{0, 1\}^\lambda$. Since the adversary is allowed to make at most a polynomial number of queries, with all but negligible

probability the prover does not sample the same m for two different proof queries. In that case, for every pk^m that is sampled respective to crs'_λ , the prover generates at most a single proof.

We therefore can simulate those proofs with the single-statement zero-knowledge simulator of 1DPNIZK. Formally, we change the way that values of the form pk^m and π^m are generated:

- Change the way that pk^m is generated during the $(\lambda - 1)$ th iteration of Step (3) of Prove:

$$(\text{pk}^m, \tau^m) \leftarrow \text{1DPNIZK.S}_1(\text{crs}'_\lambda) .$$

- Change the way that π^m is generated during Step (4) of Prove:

$$\pi^m \leftarrow \text{1DPNIZK.S}_2(\text{crs}'_\lambda, C, \text{pk}^m, \tau^m, x) .$$

Hybrids $\mathcal{H}_{\lambda-1,3}$ and \mathcal{H}_λ are indistinguishable due to the single-statement zero-knowledge of 1DPNIZK respective to crs'_λ . This hybrid is identical to the simulator, which completes the proof. \square

References

- [BF11a] Dan Boneh and David Mandell Freeman. Homomorphic signatures for polynomial functions. In Kenneth G. Paterson, editor, *Advances in Cryptology - EUROCRYPT 2011 - 30th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Tallinn, Estonia, May 15-19, 2011. Proceedings*, volume 6632 of *Lecture Notes in Computer Science*, pages 149–168. Springer, 2011.
- [BF11b] Dan Boneh and David Mandell Freeman. Linearly homomorphic signatures over binary fields and new tools for lattice-based signatures. In Dario Catalano, Nelly Fazio, Rosario Gennaro, and Antonio Nicolosi, editors, *Public Key Cryptography - PKC 2011 - 14th International Conference on Practice and Theory in Public Key Cryptography, Taormina, Italy, March 6-9, 2011. Proceedings*, volume 6571 of *Lecture Notes in Computer Science*, pages 1–16. Springer, 2011.
- [BF14] Mihir Bellare and Georg Fuchsbauer. Policy-based signatures. In Krawczyk [Kra14], pages 520–537.
- [BFKW09] Dan Boneh, David Mandell Freeman, Jonathan Katz, and Brent Waters. Signing a linear subspace: Signature schemes for network coding. In Stanislaw Jarecki and Gene Tsudik, editors, *Public Key Cryptography - PKC 2009, 12th International Conference on Practice and Theory in Public Key Cryptography, Irvine, CA, USA, March 18-20, 2009. Proceedings*, volume 5443 of *Lecture Notes in Computer Science*, pages 68–87. Springer, 2009.
- [BFM88] Manuel Blum, Paul Feldman, and Silvio Micali. Non-interactive zero-knowledge and its applications (extended abstract). In Janos Simon, editor, *Proceedings of the 20th Annual ACM Symposium on Theory of Computing, May 2-4, 1988, Chicago, Illinois, USA*, pages 103–112. ACM, 1988.

- [BGI13] Elette Boyle, Shafi Goldwasser, and Ioana Ivan. Functional signatures and pseudorandom functions. In Krawczyk [Kra14], pages 501–519. IACR ePrint 2013 <http://eprint.iacr.org/2013/401>.
- [BV14] Zvika Brakerski and Vinod Vaikuntanathan. Lattice-based FHE as secure as PKE. In Moni Naor, editor, *Innovations in Theoretical Computer Science, ITCS'14, Princeton, NJ, USA, January 12-14, 2014*, pages 1–12. ACM, 2014.
- [BZ14] Dan Boneh and Mark Zhandry. Multiparty key exchange, efficient traitor tracing, and more from indistinguishability obfuscation. In Juan A. Garay and Rosario Gennaro, editors, *Advances in Cryptology - CRYPTO 2014 - 34th Annual Cryptology Conference, Santa Barbara, CA, USA, August 17-21, 2014, Proceedings, Part I*, volume 8616 of *Lecture Notes in Computer Science*, pages 480–499. Springer, 2014.
- [CCH⁺19] Ran Canetti, Yilei Chen, Justin Holmgren, Alex Lombardi, Guy N. Rothblum, Ron D. Rothblum, and Daniel Wichs. Fiat-shamir: from practice to theory. In Moses Charikar and Edith Cohen, editors, *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019*, pages 1082–1090. ACM, 2019.
- [CJL09] Denis Xavier Charles, Kamal Jain, and Kristin E. Lauter. Signatures for network coding. *IJCoT*, 1(1):3–14, 2009.
- [Dam92] Ivan Damgård. Non-interactive circuit based proofs and non-interactive perfect zero-knowledge with preprocessing. In Rainer A. Rueppel, editor, *Advances in Cryptology - EUROCRYPT '92, Workshop on the Theory and Application of Cryptographic Techniques, Balatonfüred, Hungary, May 24-28, 1992, Proceedings*, volume 658 of *Lecture Notes in Computer Science*, pages 341–355. Springer, 1992.
- [FLS90] Uriel Feige, Dror Lapidot, and Adi Shamir. Multiple non-interactive zero knowledge proofs based on a single random string (extended abstract). In *31st Annual Symposium on Foundations of Computer Science, St. Louis, Missouri, USA, October 22-24, 1990, Volume I*, pages 308–317. IEEE Computer Society, 1990.
- [Gen09] Craig Gentry. *A fully homomorphic encryption scheme*. PhD thesis, Stanford University, 2009. crypto.stanford.edu/craig.
- [GGI⁺15] Craig Gentry, Jens Groth, Yuval Ishai, Chris Peikert, Amit Sahai, and Adam D. Smith. Using fully homomorphic hybrid encryption to minimize non-interactive zero-knowledge proofs. *J. Cryptology*, 28(4):820–843, 2015.
- [GKKR10] Rosario Gennaro, Jonathan Katz, Hugo Krawczyk, and Tal Rabin. Secure network coding over the integers. In Phong Q. Nguyen and David Pointcheval, editors, *Public Key Cryptography - PKC 2010, 13th International Conference on Practice and Theory in Public Key Cryptography, Paris, France, May 26-28, 2010. Proceedings*, volume 6056 of *Lecture Notes in Computer Science*, pages 142–160. Springer, 2010.
- [GVW15] Sergey Gorbunov, Vinod Vaikuntanathan, and Daniel Wichs. Leveled fully homomorphic signatures from standard lattices. In Rocco A. Servedio and Ronitt Rubinfeld, editors, *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, June 14-17, 2015*, pages 469–477. ACM, 2015.
- [KNYY19] Shuichi Katsumata, Ryo Nishimaki, Shota Yamada, and Takashi Yamakawa. Designated verifier/prover and preprocessing nizks from diffie-hellman assumptions. In Yuval Ishai and Vincent Rijmen, editors, *Advances in Cryptology - EUROCRYPT 2019 - 38th Annual International*

- Conference on the Theory and Applications of Cryptographic Techniques, Darmstadt, Germany, May 19-23, 2019, Proceedings, Part II*, volume 11477 of *Lecture Notes in Computer Science*, pages 622–651. Springer, 2019.
- [KNYY20] Shuichi Katsumata, Ryo Nishimaki, Shota Yamada, and Takashi Yamakawa. Compact nizks from standard assumptions on bilinear maps. In Anne Canteaut and Yuval Ishai, editors, *Advances in Cryptology - EURO-CRYPT 2020 - 39th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Zagreb, Croatia, May 10-14, 2020, Proceedings, Part III*, volume 12107 of *Lecture Notes in Computer Science*, pages 379–409. Springer, 2020.
- [KP95] Joe Kilian and Erez Petrank. An efficient non-interactive zero-knowledge proof system for NP with general assumptions. *Electron. Colloquium Comput. Complex.*, 2(38), 1995.
- [Kra14] Hugo Krawczyk, editor. *Public-Key Cryptography - PKC 2014 - 17th International Conference on Practice and Theory in Public-Key Cryptography, Buenos Aires, Argentina, March 26-28, 2014. Proceedings*, volume 8383 of *Lecture Notes in Computer Science*. Springer, 2014.
- [KW18] Sam Kim and David J. Wu. Multi-theorem preprocessing nizks from lattices. In Hovav Shacham and Alexandra Boldyreva, editors, *Advances in Cryptology - CRYPTO 2018 - 38th Annual International Cryptology Conference, Santa Barbara, CA, USA, August 19-23, 2018, Proceedings, Part II*, volume 10992 of *Lecture Notes in Computer Science*, pages 733–765. Springer, 2018.
- [MPR11] Hemanta K. Maji, Manoj Prabhakaran, and Mike Rosulek. Attribute-based signatures. In Aggelos Kiayias, editor, *Topics in Cryptology - CT-RSA 2011 - The Cryptographers' Track at the RSA Conference 2011, San Francisco, CA, USA, February 14-18, 2011. Proceedings*, volume 6558 of *Lecture Notes in Computer Science*, pages 376–392. Springer, 2011.
- [PS19] Chris Peikert and Sina Shiehian. Noninteractive zero knowledge for NP from (plain) learning with errors. In Alexandra Boldyreva and Daniele Micciancio, editors, *Advances in Cryptology - CRYPTO 2019 - 39th Annual International Cryptology Conference, Santa Barbara, CA, USA, August 18-22, 2019, Proceedings, Part I*, volume 11692 of *Lecture Notes in Computer Science*, pages 89–114. Springer, 2019.
- [Reg05] Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. In Harold N. Gabow and Ronald Fagin, editors, *STOC*, pages 84–93. ACM, 2005. Full version in [Reg09].
- [Reg09] Oded Regev. On lattices, learning with errors, random linear codes, and cryptography. *J. ACM*, 56(6), 2009.
- [SAH16] Yusuke Sakai, Nuttapong Attrapadung, and Goichiro Hanaoka. Attribute-based signatures for circuits from bilinear map. In Chen-Mou Cheng, Kai-Min Chung, Giuseppe Persiano, and Bo-Yin Yang, editors, *Public-Key Cryptography - PKC 2016 - 19th IACR International Conference on Practice and Theory in Public-Key Cryptography, Taipei, Taiwan, March 6-9, 2016, Proceedings, Part I*, volume 9614 of *Lecture Notes in Computer Science*, pages 283–300. Springer, 2016.
- [SKAH18] Yusuke Sakai, Shuichi Katsumata, Nuttapong Attrapadung, and Goichiro Hanaoka. Attribute-based signatures for unbounded languages from standard assumptions. In Thomas Peyrin and Steven D. Galbraith, editors,

- Advances in Cryptology - ASIACRYPT 2018 - 24th International Conference on the Theory and Application of Cryptology and Information Security, Brisbane, QLD, Australia, December 2-6, 2018, Proceedings, Part II*, volume 11273 of *Lecture Notes in Computer Science*, pages 493–522. Springer, 2018.
- [SMP87] Alfredo De Santis, Silvio Micali, and Giuseppe Persiano. Non-interactive zero-knowledge proof systems. In Carl Pomerance, editor, *Advances in Cryptology - CRYPTO '87, A Conference on the Theory and Applications of Cryptographic Techniques, Santa Barbara, California, USA, August 16-20, 1987, Proceedings*, volume 293 of *Lecture Notes in Computer Science*, pages 52–72. Springer, 1987.
- [Tsa17] Rotem Tsabary. An equivalence between attribute-based signatures and homomorphic signatures, and new constructions for both. *IACR Cryptology ePrint Archive*, 2017:723, 2017.