

# From Non-Adaptive to Adaptive Pseudorandom Functions

Itay Berman and Iftach Haitner\*

School of Computer Science, Tel Aviv University  
{itayberm@post.tau.ac.il, iftachh@cs.tau.ac.il}

**Abstract.** Unlike the standard notion of pseudorandom functions (PRF), a *non-adaptive* PRF is only required to be indistinguishable from random in the eyes of a *non-adaptive* distinguisher (i.e., one that prepares its oracle calls in advance). A recent line of research has studied the possibility of a *direct* construction of adaptive PRFs from non-adaptive ones, where direct means that the constructed adaptive PRF uses only few (ideally, constant number of) calls to the underlying non-adaptive PRF. Unfortunately, this study has only yielded negative results, showing that “natural” such constructions are unlikely to exist (e.g., Myers [EUROCRYPT ’04], Pietrzak [CRYPTO ’05, EUROCRYPT ’06]). We give an affirmative answer to the above question, presenting a direct construction of adaptive PRFs from non-adaptive ones. Our construction is extremely simple, a composition of the non-adaptive PRF with an appropriate pairwise independent hash function.

## 1 Introduction

A pseudorandom function family (PRF), introduced by Goldreich, Goldwasser, and Micali [11], cannot be distinguished from a family of *truly* random functions by an efficient distinguisher who is given an oracle access to a random member of the family. PRFs have an extremely important role in cryptography, allowing parties, which share a common secret key, to send secure messages, identify themselves and to authenticate messages [10, 13]. In addition, they have many other applications, essentially in any setting that requires random function provided as black-box [2, 3, 6, 7, 14, 18]. Different PRF constructions are known in the literature, whose security is based on different hardness assumption. Constructions relevant to this work are those based on the existence of pseudorandom generators [11] (and thus on the existence of one-way functions [12]), and on, the so called, synthesizers [17].

In this work we study the question of constructing (adaptive) PRFs from *non-adaptive* PRFs. The latter primitive is a (weaker) variant of the standard PRF we mentioned above, whose security is only guaranteed to hold against non-adaptive distinguishers (i.e., ones that “write” all their queries before the

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first oracle call). Since a non-adaptive PRF can be easily cast as a pseudorandom generator or as a synthesizer, [11, 17] tell us how to construct (adaptive) PRF from a non-adaptive one. In both of these constructions, however, the resulting (adaptive) PRF makes  $\Theta(n)$  calls to the underlying non-adaptive PRF (where  $n$  being the input length of the functions).<sup>1</sup>

A recent line of work has tried to figure out whether more efficient reductions from adaptive to non-adaptive PRF's are likely to exist. In a sequence of works [16, 19, 20, 5], it was shown that several “natural” approaches (e.g., composition or XORing members of the non-adaptive family with itself) are unlikely to work. See more in Section 1.3.

### 1.1 Our Result

We show that a simple composition of a non-adaptive PRF with an appropriate pairwise independent hash function, yields an adaptive PRF. To state our result more formally, we use the following definitions: a function family  $\mathcal{F}$  is  $T = T(n)$ -adaptive PRF, if no distinguisher of running time at most  $T$ , can tell a random member of  $\mathcal{F}$  from a random function with advantage larger than  $1/T$ . The family  $\mathcal{F}$  is  $T$ -non-adaptive PRF, if the above is only guarantee to hold against non-adaptive distinguishers. Given two function families  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , we let  $\mathcal{F}_1 \circ \mathcal{F}_2$  [resp.,  $\mathcal{F}_1 \oplus \mathcal{F}_2$ ] be the function family whose members are all pairs  $(f, g) \in \mathcal{F}_1 \times \mathcal{F}_2$ , and the action  $(f, g)(x)$  is defined as  $f(g(x))$  [resp.,  $f(x) \oplus g(x)$ ]. We prove the following statements (see Section 3 for the formal statements).

**Theorem 1 (Informal).** *Let  $\mathcal{F}$  be a  $(p(n) \cdot T(n))$ -non-adaptive PRF, where  $p \in \text{poly}$  is function of the evaluating time of  $\mathcal{F}$ , and let  $\mathcal{H}$  be an efficient pairwise-independent function family mapping strings of length  $n$  to  $[T(n)]_{\{0,1\}^n}$ , where  $[T]_{\{0,1\}^n}$  is the first  $T$  elements (in lexicographic order) of  $\{0,1\}^n$ . Then  $\mathcal{F} \circ \mathcal{H}$  is a  $(\sqrt[3]{T(n)}/2)$ -adaptive PRF.*

For instance, assuming that  $\mathcal{F}$  is a  $(p(n) \cdot 2^{cn})$ -non-adaptive PRF and that  $\mathcal{H}$  maps strings of length  $n$  to  $[2^{cn}]_{\{0,1\}^n}$ , Theorem 1 yields that  $\mathcal{F} \circ \mathcal{H}$  is a  $(2^{\frac{cn}{3}} - 1)$ -adaptive PRF.

Theorem 1 is only useful, however, for polynomial-time computable  $T$ 's (in this case, the family  $\mathcal{H}$  assumed by the theorem exists, see Definition 3). Unfortunately, in the important case where  $\mathcal{F}$  is only assumed to be polynomially secure non-adaptive PRF, no useful polynomial-time computable  $T$  is guaranteed to exist.<sup>2</sup>

We suggest two different solutions for handling polynomially secure PRFs. In Section 4 we observe (following Bellare [1]) that a polynomially secure non-adaptive PRF is a  $T$ -non-adaptive PRF for some  $T \in n^{\omega(1)}$ . Since this  $T$  can

<sup>1</sup> We remark that if one is only interested in *polynomial security* (i.e., no adaptive PPT distinguishes with more than negligible probability), then  $w(\log n)$  calls are sufficient (cf., [8, Sec. 3.8.4, Exe. 30]).

<sup>2</sup> Clearly  $\mathcal{F}$  is  $p$ -non-adaptive PRF for any  $p \in \text{poly}$ , but applying Theorem 1 with  $T \in \text{poly}$ , does not yield a polynomially secure adaptive PRF.

be assumed without loss of generality to be a power of two, Theorem 1 yields a non-uniform (uses  $n$ -bit advice) polynomially secure adaptive PRF, that makes a single call to the underlying non-adaptive PRF. Our second solution is to use the following “combiner”, to construct a (uniform) adaptively secure PRF, which makes  $\omega(1)$  parallel calls to the underlying non-adaptive PRF.

**Corollary 1 (Informal).** *Let  $\mathcal{F}$  be a polynomially secure non-adaptive PRF, let  $\mathcal{H} = \{\mathcal{H}_n\}_{n \in \mathbb{N}}$  be an efficient pairwise-independent length-preserving function family and let  $k(n) \in \omega(1)$  be polynomial-time computable function.*

*For  $n \in \mathbb{N}$  and  $i \in [n]$ , let  $\widehat{\mathcal{H}}_n^i$  be the function family  $\widehat{\mathcal{H}}_n^i = \{\widehat{h} : h \in \mathcal{H}\}$ , where  $\widehat{h}(x) = 0^{n-i} \|h(x)_{1,\dots,i}\|$  ( $\|\cdot\|$  stands for string concatenation). Then the ensemble  $\{\bigoplus_{i \in [k(n)]} (\mathcal{F}_n \circ \widehat{\mathcal{H}}_n^{[i \cdot \log n]})\}_{n \in \mathbb{N}}$  is a polynomially secure adaptive PRF.*

## 1.2 Proof Idea

To prove Theorem 1 we first show that  $\mathcal{F} \circ \mathcal{H}$  is indistinguishable from  $\Pi \circ \mathcal{H}$ , where  $\Pi$  being the set of *all* functions from  $\{0, 1\}^n$  to  $\{0, 1\}^{\ell(n)}$  (letting  $\ell(n)$  be  $\mathcal{F}$ 's output length), and then conclude the proof by showing that  $\Pi \circ \mathcal{H}$  is indistinguishable from  $\Pi$ .

**$\mathcal{F} \circ \mathcal{H}$  is indistinguishable from  $\Pi \circ \mathcal{H}$ .** Let  $D$  be (a possibly adaptive) algorithm of running time  $T(n)$ , which distinguishes  $\mathcal{F} \circ \mathcal{H}$  from  $\Pi \circ \mathcal{H}$  with advantage  $\varepsilon(n)$ . We use  $D$  to build a *non-adaptive* distinguisher  $\widehat{D}$  of running time  $p(n) \cdot T(n)$ , which distinguishes  $\mathcal{F}$  from  $\Pi$  with advantage  $\varepsilon(n)$ . Given an oracle access to a function  $\phi$ , the distinguisher  $\widehat{D}^\phi(1^n)$  first queries  $\phi$  on *all* the elements of  $[T(n)]_{\{0,1\}^n}$ . Next it chooses at uniform  $h \in \mathcal{H}$ , and uses the stored answers to its queries, to emulate  $D^{\phi \circ h}(1^n)$ .

Since  $\widehat{D}$  runs in time  $p(n) \cdot T(n)$ , for some large enough  $p \in \text{poly}$ , makes *non-adaptive* queries, and distinguishes  $\mathcal{F}$  from  $\Pi$  with advantage  $\varepsilon(n)$ , the assumed security of  $\mathcal{F}$  yields that  $\varepsilon(n) < \frac{1}{p(n) \cdot T(n)}$ .

**$\Pi \circ \mathcal{H}$  is indistinguishable from  $\Pi$ .** We prove that  $\Pi \circ \mathcal{H}$  is *statistically* indistinguishable from  $\Pi$ . Namely, even an unbounded distinguisher (that makes bounded number of calls) cannot distinguish between the families. The idea of the proof is fairly simple. Let  $D$  be an  $s$ -query algorithm trying to distinguish between  $\Pi \circ \mathcal{H}$  and  $\Pi$ . We first note that the distinguishing advantage of  $D$  is bounded by its probability of finding a collision in a random  $\phi \in \Pi \circ \mathcal{H}$  (in case no collision occurs,  $\phi$ 's output is uniform). We next argue that in order to find a collision in  $\phi$ , the distinguisher  $D$  gains nothing from being adaptive. Indeed, assuming that  $D$  found no collision until the  $i$ 'th call, then it has only learned that  $h$  does not collide on these first  $i$  queries. Therefore, a random (or even a constant) query as the  $(i+1)$  call, has the same chance to yield a collision, as any other query has. Hence, we assume without loss of generality that  $D$  is non-adaptive, and use the

pairwise independence of  $\mathcal{H}$  to conclude that  $D$ 's probability in finding a collision, and thus its distinguishing advantage, is bounded by  $s(n)^2/T(n)$ .

Combining the above two observations, we conclude that an adaptive distinguisher whose running time is bounded by  $\frac{1}{2}\sqrt[3]{T(n)}$ , cannot distinguish  $\mathcal{F} \circ \mathcal{H}$  from  $\Pi$  (i.e., from a random function) with an advantage better than  $\frac{T(n)^{\frac{3}{8}}}{T(n)} + \frac{1}{p(n)T(n)} \leq 2/\sqrt[3]{T(n)}$ . Namely,  $\mathcal{F} \circ \mathcal{H}$  is a  $(\sqrt[3]{T(n)}/2)$ -adaptive PRF.

### 1.3 Related Work

Maurer and Pietrzak [15] were the first to consider the question of building adaptive PRFs from non-adaptive ones. They showed that in the *information theoretic* model, a self composition of a non-adaptive PRF *does* yield an adaptive PRF.<sup>3</sup>

In contrast, the situation in the *computational model* (which we consider here) seems very different: Myers [16] proved that it is impossible to reprove the result of [15] via fully-black-box reductions. Pietrzak [19] showed that under the Decisional Diffie-Hellman (DDH) assumption, composition does not imply adaptive security. Where in [20] he showed that the existence of non-adaptive PRFs whose composition is not adaptively secure, yields that key-agreement protocol exists. Finally, Cho et al. [5] generalized [20] by proving that composition of two non-adaptive PRFs is not adaptively secure, iff (uniform transcript) key agreement protocol exists. We mention that [16, 19, 5], and in a sense also [15], hold also with respect to XORing of the non-adaptive families.

## 2 Preliminaries

### 2.1 Notations

All logarithms considered here are in base two. We let ‘||’ denote string concatenation. We use calligraphic letters to denote sets, uppercase for random variables, and lowercase for values. For an integer  $t$ , we let  $[t] = \{1, \dots, t\}$ , and for a set  $\mathcal{S} \subseteq \{0, 1\}^*$  with  $|\mathcal{S}| \geq t$ , we let  $[t]_{\mathcal{S}}$  be the first  $t$  elements (in increasing lexicographic order) of  $\mathcal{S}$ . A function  $\mu: \mathbb{N} \rightarrow [0, 1]$  is *negligible*, denoted  $\mu(n) = \text{neg}(n)$ , if  $\mu(n) = n^{-\omega(1)}$ . We let  $\text{poly}$  denote the set all polynomials, and let  $\text{PPT}$  denote the set of probabilistic algorithms (i.e., Turing machines) that run in *strictly* polynomial time.

Given a random variable  $X$ , we write  $X(x)$  to denote  $\Pr[X = x]$ , and write  $x \leftarrow X$  to indicate that  $x$  is selected according to  $X$ . Similarly, given a finite set  $\mathcal{S}$ , we let  $s \leftarrow \mathcal{S}$  denote that  $s$  is selected according to the uniform distribution on  $\mathcal{S}$ . The *statistical distance* of two distributions  $P$  and  $Q$  over a finite set  $\mathcal{U}$ , denoted as  $\text{SD}(P, Q)$ , is defined as  $\max_{\mathcal{S} \subseteq \mathcal{U}} |P(\mathcal{S}) - Q(\mathcal{S})| = \frac{1}{2} \sum_{u \in \mathcal{U}} |P(u) - Q(u)|$ .

<sup>3</sup> Specifically, assuming that the non-adaptive PRF is  $(Q, \varepsilon)$ -non-adaptively secure, no  $Q$ -query non-adaptive algorithm distinguishes it from random with advantage larger than  $\varepsilon$ , then the resulting PRF is  $(Q, \varepsilon(1 + \ln \frac{1}{\varepsilon}))$ -adaptively secure.

## 2.2 Ensemble of Function Families

Let  $\mathcal{F} = \{\mathcal{F}_n: \mathcal{D}_n \mapsto \mathcal{R}_n\}_{n \in \mathbb{N}}$  stands for an ensemble of function families, where each  $f \in \mathcal{F}_n$  has domain  $\mathcal{D}_n$  and its range contained in  $\mathcal{R}_n$ . Such ensemble is *length preserving*, if  $\mathcal{D}_n = \mathcal{R}_n = \{0, 1\}^n$  for every  $n$ .

**Definition 1 (efficient function family ensembles).** *A function family ensemble  $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{N}}$  is efficient, if the following hold:*

**Samplable.**  *$\mathcal{F}$  is samplable in polynomial-time: there exists a PPT that given  $1^n$ , outputs (the description of) a uniform element in  $\mathcal{F}_n$ .*

**Efficient.** *There exists a polynomial-time algorithm that given  $x \in \{0, 1\}^n$  and (a description of)  $f \in \mathcal{F}_n$ , outputs  $f(x)$ .*

### Operating on Function Families

**Definition 2 (composition of function families).** *Let  $\mathcal{F}^1 = \{\mathcal{F}_n^1: \mathcal{D}_n^1 \mapsto \mathcal{R}_n^1\}_{n \in \mathbb{N}}$  and  $\mathcal{F}^2 = \{\mathcal{F}_n^2: \mathcal{D}_n^2 \mapsto \mathcal{R}_n^2\}_{n \in \mathbb{N}}$  be two ensembles of function families with  $\mathcal{R}_n^1 \subseteq \mathcal{D}_n^2$  for every  $n$ . We define the composition of  $\mathcal{F}^1$  with  $\mathcal{F}^2$  as  $\mathcal{F}^2 \circ \mathcal{F}^1 = \{\mathcal{F}_n^2 \circ \mathcal{F}_n^1: \mathcal{D}_n^1 \mapsto \mathcal{R}_n^2\}_{n \in \mathbb{N}}$ , where  $\mathcal{F}_n^2 \circ \mathcal{F}_n^1 = \{(f_2, f_1) \in \mathcal{F}_n^2 \times \mathcal{F}_n^1\}$ , and  $(f_2, f_1)(x) := f_2(f_1(x))$ .*

**Definition 3 (XOR of function families).** *Let  $\mathcal{F}^1 = \{\mathcal{F}_n^1: \mathcal{D}_n^1 \mapsto \mathcal{R}_n^1\}_{n \in \mathbb{N}}$  and  $\mathcal{F}^2 = \{\mathcal{F}_n^2: \mathcal{D}_n^2 \mapsto \mathcal{R}_n^2\}_{n \in \mathbb{N}}$  be two ensembles of function families with  $\mathcal{R}_n^1, \mathcal{R}_n^2 \subseteq \{0, 1\}^{\ell(n)}$  for every  $n$ . We define the XOR of  $\mathcal{F}^1$  with  $\mathcal{F}^2$  as  $\mathcal{F}^2 \oplus \mathcal{F}^1 = \{\mathcal{F}_n^2 \oplus \mathcal{F}_n^1: \mathcal{D}_n^1 \cap \mathcal{D}_n^2 \mapsto \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$ , where  $\mathcal{F}_n^2 \oplus \mathcal{F}_n^1 = \{(f_2, f_1) \in \mathcal{F}_n^2 \times \mathcal{F}_n^1\}$ , and  $(f_2, f_1)(x) := f_2(x) \oplus f_1(x)$ .*

### Pairwise Independent Hashing

**Definition 4 (pairwise independent families).** *A function family  $\mathcal{H} = \{h: \mathcal{D} \mapsto \mathcal{R}\}$  is pairwise independent (with respect to  $\mathcal{D}$  and  $\mathcal{R}$ ), if*

$$\Pr_{h \leftarrow \mathcal{H}}[h(x_1) = y_1 \wedge h(x_2) = y_2] = \frac{1}{|\mathcal{R}|^2},$$

for every distinct  $x_1, x_2 \in \mathcal{D}$  and every  $y_1, y_2 \in \mathcal{R}$ .

For every  $\ell \in \text{poly}$ , the existence of efficient pairwise-independent family ensembles mapping strings of length  $n$  to strings of length  $\ell(n)$  is well known ([4]). In this paper we use efficient pairwise-independent function family ensembles mapping strings of length  $n$  to the set  $[T(n)]_{\{0,1\}^n}$ , where  $T(n) \leq 2^n$  and is without loss of generality a power of two.<sup>4</sup> Let  $\mathcal{H}$  be an efficient length-preserving, pairwise-independent function family ensemble and assume that  $t(n) := \log T(n)$  is polynomial-time computable. Then the function family  $\widehat{\mathcal{H}} = \{\widehat{\mathcal{H}}_n = \{h': h \in \mathcal{H}_n, h'(x) = 0^{n-t(n)} \| h(x)_{1,\dots,t(n)}\}\}$ , is an efficient pairwise-independent function family ensemble, mapping strings of length  $n$  to the set  $[T(n)]_{\{0,1\}^n}$ .

<sup>4</sup> For our applications, see Section 3, we can always consider  $T'(n) = 2^{\lceil \log(T(n)) \rceil}$ , which only causes us a factor of two loss in the resulting security.

### Pseudorandom Functions

**Definition 5 (pseudorandom functions).** An efficient function family ensemble  $\mathcal{F} = \{\mathcal{F}_n: \{0, 1\}^n \mapsto \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$  is a  $(T(n), \varepsilon(n))$ -adaptive PRF, if for every oracle-aided algorithm (distinguisher)  $\mathsf{D}$  of running time  $T(n)$  and large enough  $n$ , it holds that

$$|\Pr_{f \leftarrow \mathcal{F}_n}[\mathsf{D}^f(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[\mathsf{D}^\pi(1^n) = 1]| \leq \varepsilon(n),$$

where  $\Pi_n$  is the set of all functions from  $\{0, 1\}^n$  to  $\{0, 1\}^{\ell(n)}$ . If we limit  $\mathsf{D}$  above to be non-adaptive (i.e., it has to write all his oracle calls before making the first call), then  $\mathcal{F}$  is called  $(T(n), \varepsilon(n))$ -non-adaptive PRF.

The ensemble  $\mathcal{F}$  is a  $t$ -adaptive PRF, if it is a  $(t, 1/t)$ -adaptive PRF according to the above definition. It is polynomially secure adaptive PRF (for short, adaptive PRF), if it is a  $p$ -adaptive PRF for every  $p \in \text{poly}$ . Finally, it is super-polynomial secure adaptive PRF, if it  $T$ -adaptive PRF for some  $T(n) \in n^{\omega(1)}$ . The same conventions are also used for non-adaptive PRFs.

Clearly, a super-polynomial secure PRF is also polynomially secure. In Section 4 we prove that the converse is also true: a polynomially secure PRF is also super-polynomial secure PRF.

### 3 Our Construction

In this section we present the main contribution of this paper — a direct construction of an adaptive pseudorandom function family from a non-adaptive one.

**Theorem 2 (restatement of Theorem 1).** Let  $T$  be a polynomial-time computable integer function, let  $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^n \mapsto [T(n)]_{\{0, 1\}^n}\}$  be an efficient pairwise independent function family ensemble, and let  $\mathcal{F} = \{\mathcal{F}_n: \{0, 1\}^n \mapsto \{0, 1\}^{\ell(n)}\}$  be a  $(p(n) \cdot T(n), \varepsilon(n))$ -non-adaptive PRF, where  $p \in \text{poly}$  is determined by the computation time of  $T$ ,  $\mathcal{F}$  and  $\mathcal{H}$ . Then  $\mathcal{F} \circ \mathcal{H}$  is a  $(s(n), \varepsilon(n) + \frac{s(n)^2}{T(n)})$ -adaptive PRF for every  $s(n) < T(n)$ .

Theorem 2 yields the following simpler statement.

**Corollary 2.** Let  $T$ ,  $p$  and  $\mathcal{H}$  be as in Theorem 2. Assuming  $\mathcal{F}$  is a  $(p(n)T(n))$ -non-adaptive PRF, then  $\mathcal{F} \circ \mathcal{H}$  is a  $(\sqrt[3]{T(n)}/2)$ -adaptive PRF.

*Proof.* Applying Theorem 2 with respect to  $s(n) = \sqrt[3]{T(n)}/2$  and  $\varepsilon(n) = \frac{1}{p(n)T(n)}$ , yields that  $\mathcal{F} \circ \mathcal{H}$  is a  $(s(n), \frac{1}{p(n)T(n)} + \frac{s(n)^2}{T(n)})$ -adaptive PRF. Since  $\frac{1}{p(n)T(n)} < \frac{1}{2s(n)}$  and  $\frac{s(n)^2}{T(n)} \leq \frac{1}{2s(n)}$ , it follows that  $\mathcal{F} \circ \mathcal{H}$  is a  $(s, 1/s)$ -adaptive PRF.  $\square$

To prove Theorem 2, we use the (non efficient) function family ensemble  $\Pi \circ \mathcal{H}$ , where  $\Pi = \Pi_\ell$  (i.e., the ensemble of all functions from  $\{0, 1\}^n$  to  $\{0, 1\}^\ell$ ), and  $\ell = \ell(n)$  is the output length of  $\mathcal{F}$ . We first show that  $\mathcal{F} \circ \mathcal{H}$  is *computationally* indistinguishable from  $\Pi \circ \mathcal{H}$ , and complete the proof showing that  $\Pi \circ \mathcal{H}$  is *statistically* indistinguishable from  $\Pi$ .

### 3.1 $\mathcal{F} \circ \mathcal{H}$ is Computationally Indistinguishable From $\Pi \circ \mathcal{H}$

**Lemma 1.** *Let  $T$ ,  $\mathcal{F}$  and  $\mathcal{H}$  be as in Theorem 2. Then for every oracle-aided distinguisher  $D$  of running time  $T$ , there exists a non-adaptive oracle-aided distinguisher  $\widehat{D}$  of running time  $p(n) \cdot T(n)$ , for some  $p \in \text{poly}$  (determined by the computation time of  $T$ ,  $\mathcal{F}$  and  $\mathcal{H}$ ), with*

$$|\Pr_{g \leftarrow \mathcal{F}_n}[\widehat{D}^g(1^n) = 1] - \Pr_{g \leftarrow \Pi_n}[\widehat{D}^g(1^n) = 1]| = |\Pr_{g \leftarrow \mathcal{F}_n \circ \mathcal{H}_n}[D^g(1^n) = 1] - \Pr_{g \leftarrow \Pi_n \circ \mathcal{H}_n}[D^g(1^n) = 1]|$$

for every  $n \in \mathbb{N}$ , where  $\Pi_n$  is the set of all functions from  $\{0, 1\}^n$  to  $\{0, 1\}^{\ell(n)}$ .

In particular, the pseudorandomness of  $\mathcal{F}$  yields that  $\mathcal{F} \circ \mathcal{H}$  is computationally indistinguishable from the ensemble  $\{\Pi_n \circ \mathcal{H}_n\}_{n \in \mathbb{N}}$  by an adaptive distinguisher of running time  $T$ .

*Proof.* The distinguisher  $\widehat{D}$  is defined as follows:

**Algorithm 3** ( $\widehat{D}$ )

**Input:**  $1^n$ .

**Oracle:** a function  $\phi$  over  $\{0, 1\}^n$ .

1. Compute  $\phi(x)$  for every  $x \in [T(n)]_{\{0,1\}^n}$ .
2. Set  $g = \phi \circ h$ , where  $h$  is uniformly chosen in  $\mathcal{H}_n$ .
3. Emulate  $D^g(1^n)$ : answer a query  $x$  to  $\phi$  made by  $D$  with  $g(x)$ , using the information obtained in Step 1.

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Note that  $\widehat{D}$  makes  $T(n)$  non-adaptive queries to  $\phi$ , and it can be implemented to run in time  $p(n)T(n)$ , for large enough  $p \in \text{poly}$ . We conclude the proof by observing that in case  $\phi$  is uniformly drawn from  $\mathcal{F}_n$ , the emulation of  $D$  done in  $\widehat{D}^\phi$  is identical to a random execution of  $D^g$  with  $g \leftarrow \mathcal{F}_n \circ \mathcal{H}_n$ . Similarly, in case  $\phi$  is uniformly drawn from  $\Pi_n$ , the emulation is identical to a random execution of  $D^\pi$  with  $\pi \leftarrow \Pi_n$ . □

### 3.2 $\Pi \circ \mathcal{H}$ is Statistically Indistinguishable From $\Pi$

The following lemma is commonly used for proving the security of hash based MACs (cf., [9, Proposition 6.3.6]), yet for completeness we give it a full proof below.

**Lemma 2.** *Let  $n, T$  be integers with  $T \leq 2^n$ , and let  $\mathcal{H}$  be a pairwise-independent function family mapping string of length  $n$  to  $[T]_{\{0,1\}^n}$ . Let  $D$  be an (unbounded)  $s$ -query oracle-aided algorithm (i.e., making at most  $s$  queries), then*

$$|\Pr_{g \leftarrow \Pi \circ \mathcal{H}}[D^g = 1] - \Pr_{\pi \leftarrow \Pi}[D^\pi = 1]| \leq s^2/T,$$

where  $\Pi$  is the set of all functions from  $\{0, 1\}^n$  to  $\{0, 1\}^\ell$  (for some  $\ell \in \mathbb{N}$ ).

*Proof.* We assume for simplicity that  $D$  is deterministic (the reduction to the randomized case is standard) and makes exactly  $s$  valid (i.e., inside  $\{0, 1\}^n$ ) distinct queries, and let  $\Omega = (\{0, 1\}^\ell)^s$ . Consider the following random process:

**Algorithm 4**

1. Emulate  $D$ , while answering the  $i$ 'th query  $q_i$  with a uniformly chosen  $a_i \in \{0, 1\}^\ell$ .  
Set  $\bar{q} = (q_1, \dots, q_s)$  and  $\bar{a} = (a_1, \dots, a_s)$ .
  2. Choose  $h \leftarrow \mathcal{H}$ .
  3. Emulate  $D$  again, while answering the  $i$ 'th query  $q'_i$  with  $a'_i = a_i$  (the same  $a_i$  from Step 1), if  $h(q'_i) \notin \{h(q'_j)\}_{j \in [i-1]}$ , and with  $a'_i = a_j$ , if  $h(q'_i) = h(q'_j)$  for some  $j \in [i-1]$ .  
Set  $\bar{q}' = (q'_1, \dots, q'_s)$  and  $\bar{a}' = (a'_1, \dots, a'_s)$ .
- .....

Let  $\bar{A}, \bar{Q}, \bar{A}', \bar{Q}'$  and  $H$  be the (jointly distributed) random variables induced by the values of  $\bar{q}, \bar{a}, \bar{q}', \bar{a}'$  and  $h$  respectively, in a random execution of the above process. It is not hard to verify that  $\bar{A}$  is distributed the same as the oracle answers in a random execution of  $D^\pi$  with  $\pi \leftarrow \Pi$ , and that  $\bar{A}'$  is distributed the same as the oracle answers in a random execution of  $D^g$  with  $g \leftarrow \Pi \circ \mathcal{H}$ . Hence, for proving Lemma 2, it suffices to bound the statistical distance between  $\bar{A}$  and  $\bar{A}'$ .

Let  $\text{Coll}$  be the event that  $H(\bar{Q}_i) = H(\bar{Q}_j)$  for some  $i \neq j \in [s]$ . Since the queries and answers in both emulations of Algorithm 4 are the same until a collision with respect to  $H$  occurs, it follows that

$$\Pr[\bar{A} \neq \bar{A}'] \leq \Pr[\text{Coll}] \tag{1}$$

On the other hand, since  $H$  is chosen *after*  $\bar{Q}$  is set, the pairwise independent of  $\mathcal{H}$  yields that

$$\Pr[\text{Coll}] \leq s^2/T, \tag{2}$$

and therefore  $\Pr[\bar{A} \neq \bar{A}'] \leq s^2/T$ . It follows that  $\Pr[\bar{A} \in C] \leq \Pr[\bar{A}' \in C] + s^2/T$  for every  $C \subseteq \Omega$ , yielding that  $\text{SD}(\bar{A}, \bar{A}') \leq s^2/T$ .  $\square$

### 3.3 Putting It Together

We are now finally ready to prove Theorem 2.

*Proof (of Theorem 2).* Let  $D$  be an oracle-aided algorithm of running time  $s$  with  $s(n) < T(n)$ . Lemma 1 yields that  $|\Pr_{g \leftarrow \mathcal{F}_n \circ \mathcal{H}_n}[D^g(1^n) = 1] - \Pr_{g \leftarrow \Pi_n \circ \mathcal{H}_n}[D^g(1^n) = 1]| \leq \varepsilon(n)$  for large enough  $n$ , where Lemma 2 yields that  $|\Pr_{g \leftarrow \Pi_n \circ \mathcal{H}_n}[D^g(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[D^\pi(1^n) = 1]| \leq s(n)^2/T(n)$  for every  $n \in \mathbb{N}$ . Hence, the triangle inequality yields that  $|\Pr_{g \leftarrow \mathcal{F}_n \circ \mathcal{H}_n}[D^g(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[D^\pi(1^n) = 1]| \leq \varepsilon(n) + s(n)^2/T(n)$  for large enough  $n$ , as requested.  $\square$



### 3.4 Handling Polynomial Security

Corollary 2 is only useful when the security of the underlying non-adaptive PRF (i.e.,  $T$ ) is efficiently computable (or when considering non-uniform PRF constructions, see Section 1.1). In this section we show how to handle the important case of polynomially secure non-adaptive PRF. We use the following “combiner”.

**Definition 6.** Let  $\mathcal{H}$  be a function family into  $\{0, 1\}^n$ . For  $i \in [n]$ , let  $\widehat{\mathcal{H}}^i$  be the function family  $\widehat{\mathcal{H}}^i = \{\widehat{h} : h \in \mathcal{H}\}$ , where  $\widehat{h}(x) = 0^{n-i} \| h(x)_{1, \dots, i}$ .

**Corollary 3.** Let  $\mathcal{F}$  be a  $T(n)$ -non-adaptive PRF, let  $\mathcal{H}$  be an efficient length-preserving pairwise-independent function family ensemble, and let  $\mathcal{I}(n) \subseteq [n]$  be polynomial-time computable (in  $n$ ) index set. Define the function family ensemble  $G = \{G_n\}_{n \in \mathbb{N}}$ , where  $G_n = \bigoplus_{i \in \mathcal{I}(n)} (\mathcal{F}_n \circ \widehat{\mathcal{H}}_n^i)$ .

There exists  $q \in \text{poly}$  such that  $G$  is a  $(\sqrt[3]{2^{t(n)}}/2)$ -adaptive PRF, for every polynomial-time computable integer function  $t$ , with  $t(n) \in \mathcal{I}(n)$  and  $2^{t(n)} \leq T(n)/q(n)$ .

Before proving the corollary, let us first use it for constructing adaptive PRF from non-adaptive polynomially secure one.

**Corollary 4 (restatement of Corollary 1).** Let  $\mathcal{F}$  be a polynomially secure non-adaptive PRF, let  $\mathcal{H}$  be an efficient pairwise-independent length-preserving function family ensemble and let  $k(n) \in \omega(1)$  be polynomial-time computable function. Then  $G := \{\bigoplus_{i \in [k(n)]} (\mathcal{F}_n \circ \widehat{\mathcal{H}}_n^{\lfloor i \cdot \log n \rfloor})\}_{n \in \mathbb{N}}$  is polynomially secure adaptive PRF.

*Proof.* Let  $\mathcal{I}(n) := \{\lfloor \log n \rfloor, \lfloor 2 \cdot \log n \rfloor, \dots, \lfloor k(n) \cdot \log n \rfloor\}$ . Applying Corollary 3 with respect to  $\mathcal{F}$ ,  $\mathcal{H}$ ,  $\mathcal{I}$  and  $t(n) = \lfloor c \cdot \log n \rfloor$ , where  $c \in \mathbb{N}$ , yields that  $G$  is a  $O(\sqrt[3]{n^c})$ -adaptive PRF. It follows that  $G$  is  $p$ -adaptive PRF for every  $p \in \text{poly}$ . Namely,  $G$  is polynomially secure adaptive PRF.  $\square$

*Remark 1 (unknown security).* Corollary 3 is also useful when the security of  $\mathcal{F}$  is “not known” in the construction time. Taking  $\mathcal{I}(n) = \{1, 2, 4, \dots, 2^{\lfloor \log n \rfloor}\}$  (resulting in  $\log n$  calls to  $\mathcal{F}$ ) and assuming that  $\mathcal{F}$  is found to be  $T(n)$ -non-adaptive PRF for some polynomial-time computable  $T$ , the resulting PRF is guaranteed to be  $O(\sqrt[6]{T(n)})$ -adaptive PRF (neglecting polynomial factors).

*Proof (of Corollary 3).* It is easy to see that  $G$  is efficient, so it is left to argue for its security. Let  $q(n) = q'(n)p(n)$ , where  $p$  is as in the statement of Corollary 2, and  $q' \in \text{poly}$  to be determined later. Let  $t$  be a polynomial-time computable integer function with  $t(n) \in \mathcal{I}(n)$  and  $2^{t(n)} \leq T(n)/q(n)$ . It follows that  $\widehat{\mathcal{H}}^t = \{\widehat{\mathcal{H}}_n^{t(n)}\}_{n \in \mathbb{N}}$  is an efficient pairwise-independent function family ensemble, and Corollary 2 yields that  $\mathcal{F} \circ \widehat{\mathcal{H}}^t$  is a  $(\sqrt[3]{q'(n)2^{t(n)}}/2)$ -adaptive PRF.

Assume towards a contradiction that there exists an oracle-aided distinguisher  $D$  that runs in time  $T'(n) = \sqrt[3]{2^{t(n)}/2}$  and

$$|\Pr_{g \leftarrow G_n}[D^g(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[D^\pi(1^n) = 1]| > 1/T'(n) \quad (3)$$

for infinitely many  $n$ 's. We use the following distinguisher for breaking the pseudorandomness of  $\mathcal{F} \circ \widehat{\mathcal{H}}^t$ :

**Algorithm 5** ( $\widehat{D}$ )

**Input:**  $1^n$ .

**Oracle:** a function  $\phi$  over  $\{0, 1\}^n$ .

1. For every  $i \in \mathcal{I}(n) \setminus \{t(n)\}$ , choose  $g^i \leftarrow \mathcal{F}_n \circ \widehat{\mathcal{H}}_n^i$ .
2. Set  $g := \phi \oplus \bigoplus_{i \in \mathcal{I}(n) \setminus \{t(n)\}} g^i$ .
3. Emulate  $D^g(1^n)$ .

.....

Note that  $\widehat{D}$  can be implemented to run in time  $|\mathcal{I}(n)| \cdot r(n) \cdot T'(n)$  for some  $r \in \text{poly}$ , which is smaller than  $\sqrt[3]{q'(n)2^{t(n)}/2}$  for large enough  $q'$ . Also note that in case  $\phi$  is uniformly distributed over  $\Pi_n$ , then  $g$  (selected by  $\widehat{D}^\phi(1^n)$ ) is uniformly distributed in  $\Pi_n$ , where in case  $\phi$  is uniformly distributed in  $\mathcal{F}_n \circ \widehat{\mathcal{H}}_n^{t(n)}$ , then  $g$  is uniformly distributed in  $G_n$ . It follows that

$$\left| \Pr_{g \leftarrow (\mathcal{F} \circ \widehat{\mathcal{H}}^t)_n}[\widehat{D}^g(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[\widehat{D}^\pi(1^n) = 1] \right| = |\Pr_{g \leftarrow G_n}[D^g(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[D^\pi(1^n) = 1]| \quad (4)$$

for every  $n \in \mathbb{N}$ . In particular, Equation (3) yields that

$$\left| \Pr_{g \leftarrow (\mathcal{F} \circ \widehat{\mathcal{H}}^t)_n}[\widehat{D}^g(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[\widehat{D}^\pi(1^n) = 1] \right| > \frac{2}{\sqrt[3]{2^{t(n)}}} > \frac{2}{\sqrt[3]{q'(n)2^{t(n)}}}$$

for infinitely many  $n$ 's, in contradiction to the pseudorandomness of  $\mathcal{F} \circ \widehat{\mathcal{H}}^t$  we proved above.  $\square$

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## 4 From Polynomial to Super-Polynomial Security

The standard security definition for cryptographic primitives is *polynomial security*: any PPT trying to break the primitive has only negligible success probability. Bellare [1] showed that for any polynomially secure primitive there exists a *single* negligible function  $\mu$ , such that no PPT can break the primitive with probability larger than  $\mu$ . Here we take his approach a step further, showing that for a polynomially secure primitive there exists a super-polynomial function  $T$ , such that no adversary of running time  $T$  breaks the primitive with probability larger than  $1/T$ .

In the following we identify algorithms with their string description. In particular, when considering algorithm  $A$ , we mean the algorithm defined by the string  $A$  (according to some canonical representation). We prove the following result.

**Theorem 6.** *Let  $v: \{0, 1\}^* \times \mathbb{N} \mapsto [0, 1]$  be a function with the following properties: 1)  $v(A, n) \leq 1/p(n)$  for every oracle-aided PPT  $A$ ,  $p \in \text{poly}$  and large enough  $n$ ; and 2) if the distributions induced by random executions of  $A^f(x)$  and  $B^f(x)$  are the same for any input  $x \in \{0, 1\}^n$  and function  $f$  (each distribution describes the algorithm's output and oracle queries), then  $v(A, n) = v(B, n)$ .*

*Then there exists an integer function  $T(n) \in n^{\omega(1)}$  such that following holds: for any algorithm  $A$  of running time at most  $T(n)$ , it holds that  $v(A, n) \leq 1/T(n)$  for large enough  $n$ .*

*Remark 2 (Applications).* Let  $f$  be a polynomially secure OWF (i.e.,  $\Pr[A(f(U_n)) \in f^{-1}(f(U_n))] = \text{neg}(n)$  for any PPT  $A$ ). Applying Theorem 6 with  $v(A, n) := \Pr[A(f(U_n)) \in f^{-1}(f(U_n))]$  (where if  $A$  expects to get an oracle, provide him with the constant function  $\phi(x) = 1$ ), yields that  $f$  is super-polynomial secure OWF (i.e., exists  $T(n) \in n^{\omega(1)}$  such that  $\Pr[A(f(U_n)) \in f^{-1}(f(U_n))] \leq 1/T(n)$  for any algorithm of running time  $T$  and large enough  $n$ ).

Similarly, for a polynomially secure PRF  $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{N}}$  (see Definition 5), applying Theorem 6 with  $v(A, n) := |\Pr_{f \leftarrow \mathcal{F}_n}[A^f(1^n) = 1] - \Pr_{\pi \leftarrow \Pi_n}[A^\pi(1^n) = 1]|$ , where  $\Pi_n$  is the set of all functions with the same domain/range as  $\mathcal{F}_n$ , yields that  $\mathcal{F}$  is super-polynomial secure PRF.

*Proof (of Theorem 6).* Given a probabilistic algorithm  $A$  and an integer  $i$ , let  $A_i$  denote the variant of  $A$  that on input of length  $n$ , halts after  $n^i$  steps (hence,  $A_i$  is a PPT). Let  $\mathcal{S}_i$  be the first  $i$  strings in  $\{0, 1\}^*$ , according to some canonical

order, viewed as descriptions of  $i$  algorithms. Let  $\mathcal{I}(n) = \{i \in [n] : \forall \mathbf{A} \in \mathcal{S}_i, k \geq n : v(\mathbf{A}_i, k) < 1/k^i\} \cup \{1\}$ , let  $t(n) = \max \mathcal{I}(n)$  and  $T(n) = n^{t(n)}$ .

Let  $\mathbf{A}$  be an algorithm of running time  $T(n)$ , and let  $i_{\mathbf{A}}$  be the first integer such that  $\mathbf{A} \in \mathcal{S}_{i_{\mathbf{A}}}$ . In Remark 2 we prove that  $t(n) \in \omega(1)$ , hence it follows that  $t(n) > i_{\mathbf{A}}$  for any large enough  $n$ . For any such  $n$ , the definition of  $t$  guarantees that  $v(\mathbf{A}_{t(n)}, n) < 1/n^{t(n)} = 1/T(n)$ . Since  $\mathbf{A}$  is of running time  $T(n)$ , the second property of  $v$  yields that  $v(\mathbf{A}, n) = v(\mathbf{A}_{t(n)}, n)$ , and therefore  $v(\mathbf{A}, n) < 1/T(n)$ .  $\square$

*Claim.* It holds that  $t(n) \in \omega(1)$ .

*Proof.* Fix  $i \in \mathbb{N}$ . For each  $\mathbf{A} \in \mathcal{S}_i$ , let  $n_{\mathbf{A}}$  be the first integer such that  $v(\mathbf{A}_i, n) \leq 1/n^i$  for every  $n \geq n_{\mathbf{A}}$  (note that such  $n_{\mathbf{A}}$  exists by the first property of  $v$ ), and let  $n_i = \max\{n_{\mathbf{A}} : \mathbf{A} \in \mathcal{S}_i\}$ . It follows that  $v(\mathbf{A}_i, n) \leq 1/n^i$  for every  $n \geq n_i$  and  $\mathbf{A} \in \mathcal{S}_i$ , and therefore  $t(n_i) \geq i$ .  $\square$