

A kilobit hidden SNFS discrete logarithm computation

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Abstract. We perform a special number field sieve discrete logarithm computation in a 1024-bit prime field. To our knowledge, this is the first kilobit-sized discrete logarithm computation ever reported for prime fields. This computation took a little over two months of calendar time on an academic cluster using the open-source CADO-NFS software.

Our chosen prime p looks random, and $p-1$ has a 160-bit prime factor, in line with recommended parameters for the Digital Signature Algorithm. However, our p has been trapdoored in such a way that the special number field sieve can be used to compute discrete logarithms in \mathbb{F}_p^* , yet detecting that p has this trapdoor seems out of reach. Twenty-five years ago, there was considerable controversy around the possibility of backdoored parameters for DSA. Our computations show that trapdoored primes are entirely feasible with current computing technology. We also describe special number field sieve discrete log computations carried out for multiple conspicuously weak primes found in use in the wild.

As can be expected from a trapdoor mechanism which we say is hard to detect, our research did not reveal any trapdoored prime in wide use. The only way for a user to defend against a hypothetical trapdoor of this kind is to require verifiably random primes.

1 Introduction

In the early 1990's, NIST published draft standards for what later became the Digital Signature Algorithm (DSA) [40]. DSA is now widely used. At the time, many members of the cryptographic community voiced concerns about the proposal. Among these concerns were that the standard encouraged the use of a global common prime modulus p [45], and that a malicious party could specially craft a trapdoored prime so that signatures would be easier to forge for the trapdoor owner [31]. This latter charge was the subject of a remarkable panel at Eurocrypt 1992 [15,1]. Most of the panelists agreed that it appeared to be difficult to construct an undetectably trapdoored modulus, and that such trapdoors appeared unlikely. To protect against possible trapdoored primes, the Digital Signature Standard suggests that primes for DSA be chosen in a “verifiably random” way, with a published seed value [49]. Yet DSA primes used in the wild today are seldom published with the seed.

Concerns about cryptographic backdoors have a long history (for instance, it has been formalized as “kleptography” in the 90’s [54]) and regained prominence in recent years since the disclosure of NSA documents leaked by Edward Snowden. A set of leaked documents published in September 2013 by the NY Times [43], The Guardian [4], and ProPublica [30] describe an NSA “SIGINT Enabling Project” that included among its goals to “Influence policies, standards, and specification for commercial public key technologies”. The newspaper articles describe documents making specific reference to a backdoor in the Dual EC random number generator, which had been standardized by both NIST and ANSI. NIST responded by withdrawing its recommendation for the Dual EC DRBG, writing “This algorithm includes default elliptic curve points for three elliptic curves, the provenance of which were not described. Security researchers have highlighted the importance of generating these elliptic curve points in a trustworthy way. This issue was identified during the development process, and the concern was initially addressed by including specifications for generating different points than the default values that were provided. However, recent community commentary has called into question the trustworthiness of these default elliptic curve points” [38]. There is evidence that the ability to backdoor the Dual EC algorithm has been exploited in the wild: Juniper Networks had implemented Dual EC in NetScreen VPN routers, but had used it with custom-generated parameters. In December 2015 Juniper published a security advisory [25] announcing that an attacker had made unauthorized modifications to the source code for these products to substitute a different curve point in the Dual EC implementation [10].

In this paper, we demonstrate that constructing and exploiting trapdoored primes for Diffie-Hellman and DSA is feasible for 1024-bit keys with modern academic computing resources. Current estimates for 1024-bit discrete log in general suggest that such computations are likely within range for an adversary who can afford hundreds of millions of dollars of special-purpose hardware [2]. In contrast, we were able to perform a discrete log computation on a specially trapdoored prime in two months on an academic cluster. While the Dual EC algorithm appears to have only rarely been used in practice [9], finite-field Diffie-Hellman and DSA are cornerstones of public-key cryptography. We neither show nor claim that trapdoored primes are currently in use. However, the near-universal failure of implementers to use verifiable prime generation practices means that use of weak primes would be undetectable in practice and unlikely to raise eyebrows.

The special number field sieve trapdoor

The Number Field Sieve (NFS), still very much in its infancy at the beginning of the 1990’s, was originally proposed as an integer factoring algorithm [32]. Gordon adapted the algorithm to compute discrete logarithms in prime fields [22]. Both for the integer factoring and the discrete logarithm variants, several theoretical and computational obstacles had to be overcome before the NFS was practical to use for large scale computations. For the past twenty years, the NFS has been routinely used in record computations, and the underlying algorithms have been

thoroughly improved. The NFS is now a versatile algorithm which can handle an arbitrary prime p , and compute discrete logarithms in \mathbb{F}_p^* in asymptotic time $L_p(1/3, (64/9)^{1/3})^{1+o(1)}$, using the usual L -notation (see §3.2).

Current computational records for the number field sieve include a 768-bit factorization of an RSA modulus, completed in December 2009 by Kleinjung et al. [27] and a 768-bit discrete log for a safe prime, completed in June 2016 by Kleinjung et al. [28].

Very early on in the development of NFS, it was observed that the algorithm was particularly efficient for inputs of a special form. Some composite integers are particularly amenable to being factored by NFS, and primes of a special form allow easier computation of discrete logarithms. This relatively rare set of inputs defines the Special Number Field Sieve (SNFS). It is straightforward to start with parameters that give a good running time for the NFS—more precisely, a pair of irreducible integer polynomials meeting certain degree and size constraints—and derive an integer to be factored, or a prime modulus for a discrete logarithm. In general, moving in the other direction, from a computational target to SNFS parameters, is known to be possible only in rare cases (e.g. the Cunningham project). The complexity of SNFS is $L_p(1/3, (32/9)^{1/3})^{1+o(1)}$, much less than its general counterpart. A 1039-bit SNFS factorization was completed in 2007 by Aoki et al. [3].

In 1992, Gordon [21] suggested several methods that are still the best known for trapdooring primes to give the best running time for the SNFS, without the trapdoored SNFS property being conspicuous³. Most of his analysis remains valid, but there has been significant improvement in the NFS algorithm in the past 25 years. In the early days, an NFS computation had to face issues of dealing with class groups and explicit units. That meant much less flexibility in creating the trapdoor, to the point that it was indeed difficult to conceal it. It is now well understood that these concerns were artificial and can be worked around [46], much to the benefit of the trapdoor designer. Gordon’s analysis and much of the discussion of trapdoored DSA primes in 1992 focused on 512-bit primes, the suggested parameter sizes for NIST’s DSS draft at the time. However, 25 years later, 1024-bit targets are of greater cryptanalytic interest.

We update the state of the art in crafting trapdoored 1024-bit primes for which practical computation of discrete logarithms is possible, and demonstrate that exploitation is practical by performing a 1024-bit SNFS discrete log computation.

We begin by reviewing in Section 2 the origin of primes found in multiple practical cryptographic contexts. Section 3 recalls a brief background on the Number Field Sieve. In Section 4, we reevaluate Gordon’s work on trapdooring primes for SNFS given the modern understanding of the algorithm, and explain for a given target size which polynomial pair yields the fastest running time. This answers a practical question for the problem at hand—how to optimally

³ In 1991, another method was suggested by Lenstra in [31], and played a role in triggering the Eurocrypt panel [15]. Gordon’s trap design is more general.

select trapdoor parameters to simplify computations with the prime—and is also of wider interest for NFS-related computations.

We then run a full 1024-bit experiment to show that trapdoored primes are indeed a practical threat, and perform a full 1024-bit SNFS discrete log computation for our trapdoored prime. We describe our computation in Section 5. We show how various adaptations to the block Wiedemann algorithm are essential to minimizing its running time, compared to previous computations of the same kind. We detail the descent procedure, and the various challenges which must be overcome so as to complete individual logs in a short time. We also provide an extensive appendix giving details on the analysis of individual logarithm computation in Appendix A, as this portion of the computation is not well detailed in the literature.

Finally, we evaluate the impact of our results in Section 6. Our computation required roughly a factor of 10 less resources than the recent 768-bit GNFS discrete log announced by Kleinjung et al. However, we have found a number of primes amenable to non-hidden SNFS DLP computations in use in the wild. We describe additional SNFS computations we performed on these primes in Section 6.2.

2 Modern security practices for discrete log cryptosystems

Verifiable prime generation. It is legitimate to wonder whether one should worry about trapdoored primes at all. Good cryptographic practice recommends that publicly agreed parameters must be “verifiably random”. For example, appendix A.1.1.2 of the FIPS 186 standard [40] proposes a method to generate DSA primes p and q from a random seed and a hash function, and suggests that one should publish that seed alongside with p and q . The publication of this seed is marked *optional*. Primes of this type are widely used for a variety of cryptographic primitives; for example NIST SP 800-56A specifies that finite-field parameters for key exchange should be generated using FIPS 186 [41, §5.5.1.1].

While it is true that some standardized cryptographic data includes “verifiable randomness”⁴ or rigidity derived from “nothing up my sleeve” numbers, it is noteworthy that this is not always the case. For example, both France and China standardized elliptic curves for public use without providing any sort of justification for the chosen parameters [8, §3.1]. RFC 5114 [33] specifies a number of groups for use with Diffie-Hellman, and states that the parameters were drawn from NIST test data, but neither the NIST test data [39] nor RFC 5114 itself contain the seeds used to generate the finite field parameters. In a similar vein, the origin of the default 2048-bit prime in the Helios voting system used in the most recent IACR Board of Directors Election in 2015 is undocumented. Most users would have to go out of their way to generate verifiable primes: the default

⁴ This still leaves the question of whether the seed is honest, see e.g. [47,8]. We do not address this concern here.

behavior of OpenSSL does not print out seeds when generating Diffie-Hellman or DSA parameter sets. However, some implementations do provide seeds. Java’s `sun.security.provider` package specifies hard-coded 512-, 768-, and 1024-bit groups together with the FIPS 186 seeds used to generate them.

Standardized and hard-coded primes. It is also legitimate to wonder whether one should be concerned about widespread reuse of primes. For modern computers, prime generation is much less computationally burdensome than in the 1990s, and any user worried about a backdoor could easily generate their own group parameters. However, even today, many applications use standardized or hard-coded primes for Diffie-Hellman and DSA. We illustrate this by several examples.

In the TLS protocol, the server specifies the group parameters that the client and server use for Diffie-Hellman key exchange. Adrian et al. [2] observed in 2015 that 37% of the Alexa Top 1 Million web sites supported a single 1024-bit group for Diffie-Hellman key exchange. The group parameters were hard-coded into Apache 2.2, without any specified seed for verification. They also observed that in May 2015, 56% of HTTPS hosts selected one of the 10 most common 1024-bit groups when negotiating ephemeral Diffie-Hellman key exchange. Among 13 million recorded TLS handshakes negotiating ephemeral Diffie-Hellman key exchange, only 68,000 distinct prime moduli were used. The TLS 1.3 draft restricts finite-field Diffie-Hellman to a set of five groups modulo safe primes ranging in size from 2048 to 8196 bits derived from the nothing-up-my-sleeve number e [19].

In the IKE protocol for IPsec, the initiator and responder negotiate a group for Diffie-Hellman key exchange from a set list of pre-defined groups; Adrian et al. observed that 66% of IKE responder hosts preferred the 1024-bit Oakley Group 2 over other choices. The Oakley groups specify a collection of primes derived from a “nothing-up-my-sleeve” number, the binary expansion of π , and have been built into standards, including IKE and SSH, for decades [42]. The additional finite-field Diffie-Hellman groups specified in RFC 5114 are widely used in practice: Internet-wide scans from September 2016 found that over 900,000 (2.25%) of TLS hosts on port 443 chose these groups [16]. Scans from February 2016 of IKE hosts on port 500 revealed that 340,000 (13%) supported the RFC 5114 finite-field Diffie-Hellman parameters [53].

RFC 4253 specifies two groups that must be supported for SSH Diffie-Hellman key exchange: Oakley Group 2 (which is referred to as SSH group 1) and Oakley Group 14, a 2048-bit prime. SSH group 1 key exchange was disabled by default in OpenSSH version 7.0, released in August 2015 [18]. Optionally, SSH clients and servers may negotiate a different group using the group exchange handshake. However, OpenSSH chooses the group negotiated during this exchange from a pre-generated list that is generally shipped with the software package. The `/etc/ssh/moduli` file on an Ubuntu 16.04 machine in our cluster contained 267 entries in size between 1535 and 8191 bits. The 40 to 50 primes at each size appear to have been generated by listing successive primes from a fixed starting point, and differ only in the least significant handful of bits. We examined data from a full IPv4 SSH scan performed in October 2015 [53] that offered Diffie-Hellman group exchange only, and found 11,658 primes in use from 10.9

million responses, many of which could be clustered into groups differing only in a handful of least significant bits.

The SSH protocol also allows servers to use long term DSA keys to authenticate themselves to clients. We conducted a scan of a random 1% portion of the IPv4 space for hosts running SSH servers on port 22 with DSA host keys in September 2016, and found that most hosts seemed to generate unique primes for their DSA public keys. The scan yielded 27,380 unique DSA host keys from 32,111 host servers, of which only 557 shared a prime with another key. DSA host key authentication was also disabled by default in OpenSSH 7.0 [18].

1024-bit primes in modern cryptographic deployments. It is well understood that 1024-bit factorization and discrete log computations are within the range of government-level adversaries [2], but such computations are widely believed by practitioners to be *only* within the range of such adversaries, and thus that these key sizes are still safe for use in many cases. While NIST has recommended a minimum prime size of 2048 bits since 2010 [6], 1024-bit primes remain extremely common in practice. Some of this is due to implementation and compatibility issues. For example, versions of Java prior to Java 8, released in 2014, did not support Diffie-Hellman or DSA group sizes larger than 1024 bits. DNSSEC limits DSA keys to a maximum size of 1024-bit keys [29], and stated, in 2012, that with respect to RSA keys, “To date, despite huge efforts, no one has broken a regular 1024-bit key; . . . it is estimated that most zones can safely use 1024-bit keys for at least the next ten years.” SSL Labs SSL Pulse estimated in September 2016 that 22% of the 200,000 most popular HTTPS web sites performed a key exchange with 1024-bit strength [50].

3 The Number Field Sieve for discrete logarithms

3.1 The NFS setting

We briefly recall the Number Field Sieve (NFS) algorithm for computing discrete logarithms in finite fields. This background is classical and can be found in a variety of references. NFS appeared first as an algorithm for factoring integers [32], and has been adapted to the computation of discrete logarithms over several works [22,23].

Let \mathbb{F}_p be a prime field, let $\gamma \in \mathbb{F}_p^*$ be an element of prime order $q \mid p - 1$. We wish to solve discrete logarithms in $\langle \gamma \rangle$. The basic outline of the NFS-DL algorithm is as follows:

Polynomial selection. Select irreducible integer polynomials f and g , sharing a common root m modulo p . Both polynomials define number fields, which we denote by $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$.

Sieving. Find many pairs a, b such that the two integers (called *norms* – albeit improperly if f or g are not monic) $\text{Res}(f(x), a - bx)$ and $\text{Res}(g(x), a - bx)$ factor completely into primes below a chosen smoothness bound B .

Filtering. Form multiplicative combinations of the (a, b) pairs to reduce the number of prime ideals appearing in the corresponding ideal factorizations.

Compute maps. Compute q -adic characters (known as Schirokauer maps [46]). This yields a relation matrix which can be written as $(M\|S)$, with M the block with ideal valuations, and S the block with Schirokauer maps.

Linear algebra. Solve the linear system $(M\|S)x = 0$, which gives *virtual logarithms*.

Individual logarithm. Given a target value $z \in \langle \gamma \rangle$, derive its logarithm as a linear combination of a subset of the virtual logarithms.

3.2 Complexity analysis

NFS complexity analysis involves the usual L -notation, defined as

$$L_p(e, c) = \exp(c(\log p)^e (\log \log p)^{1-e}). \quad (1)$$

This notation interpolates between polynomial ($e = 0$) and exponential ($e = 1$) complexities. It adapts well to working with smoothness probabilities. In this formula and elsewhere in the paper, we use the log notation for the natural logarithm. In the few places where we have formulae that involve bit sizes, we always use \log_2 for the logarithm in base 2.

The polynomials f and g have a crucial impact on the size of the integers $\text{Res}(f(x), a - bx)$ and $\text{Res}(g(x), a - bx)$, and therefore on the probability that these integers factor into primes below B (in other words, are B -smooth).

Variant	$\deg f$	$\ f\ $	$\deg g$	$\ g\ $	complexity exponent
General NFS (base- m)	d	$p^{1/(d+1)}$	1	$p^{1/(d+1)}$	$(64/9)^{1/3}$
General NFS (Joux-Lercier)	$d' + 1$	$O(1)$	d'	$p^{1/(d'+1)}$	$(64/9)^{1/3}$
Special NFS (for example)	d	$O(1)$	1	$p^{1/(d+1)}$	$(32/9)^{1/3}$

Table 1. Polynomial selection choices for NFS variants.

The analysis of the NFS depends on the prime p . When no assumptions are made on p , we have the so-called *general number field sieve* (GNFS). It is possible to perform polynomial selection so that $(\deg f, \deg g)$ is $(1, d)$ (by choosing $m \approx N^{1/d}$ and writing p in “base- m ”, or similar), or $(d' + 1, d')$ (the Joux-Lercier method [23]). The degrees d and d' in each case are integer parameters, for which an educated guess is provided by their asymptotic optimum, namely $d = (3 \log p / \log \log p)^{1/3}$, and $d' = d/2$. Both approaches lead to an overall complexity $L_p(1/3, (64/9)^{1/3})^{1+o(1)}$ for a discrete logarithm computation, as indicated in Table 1, where $\|f\|$ denotes the maximum absolute value of the coefficients of f .

In contrast, some prime numbers are such that *there exist* exceptionally small polynomial pairs (f, g) sharing a common root modulo p . This makes a considerable difference in the efficiency of the algorithm, to the point that the exponent

in the complexity drops from $(64/9)^{1/3}$ to $(32/9)^{1/3}$ —a difference which is also considerable in practice. In most previously considered cases, this special structure is clear from the number itself. For the SNFS factorization performed by Aoki et al. for the composite integer $2^{1039} - 1$, they chose $f(x) = 2x^6 - 1$ and $g(x) = x - 2^{173}$ [3].

4 Heidi hides her polynomials

Early on in the development of NFS, Gordon [21] suggested that one could craft primes so that SNFS polynomials exist, but may not be apparent to the casual observer. Heidi, a mischievous designer for a crypto standard, would select a pair of SNFS polynomials to her liking *first*, and publish only their resultant p (if it is prime) *afterwards*. The hidden trapdoor then consists in the pair of polynomials which Heidi used to generate p , and that she can use to considerably ease the computation of discrete logarithms in \mathbb{F}_p .

Twenty-five years later, we reconsider the best-case scenario for Heidi: given a target size, what type of polynomial pair will give the fastest running time for a discrete logarithm computation? For the current state of the art in algorithmic development and computation power, is there a parameter setting for which the computations are simultaneously within reach, Heidi can efficiently generate a trapdoored prime, and defeat attempts at unveiling it?

4.1 Best form for SNFS polynomials

The Special Number Field Sieve has been mostly used in the context of integer factorization, in particular for numbers from the Cunningham Project. In that case the integers are given, and typically the only way to find SNFS polynomials for these numbers is to take one linear polynomial with large coefficients and a polynomial of larger degree, with tiny coefficients. In our situation, the construction can go the opposite way: we are free to choose first the form of the polynomials, hoping that their resultant will be a prime number. We thus have more freedom in the construction of our polynomial pair.

Let n be the number of bits of the SNFS prime p we will construct. We consider first the case where we have two polynomials f and g that are non-skewed, i.e. all the coefficients of each polynomial have roughly the same size. We denote d_f and d_g their respective degrees, and $\|f\|$ and $\|g\|$ the respective maximum absolute values of their coefficients. Since the resultant must be almost equal to p , we have

$$d_f \log_2 \|g\| + d_g \log_2 \|f\| \approx n. \quad (2)$$

Let A be a bound on the a and b integers we are going to consider during relation collection. Then the product of the norms that have to be tested for smoothness can be approximated by $\|f\| \|g\| A^{d_f+d_g}$. We will try to make its size as small as possible, so we want to minimize

$$\log_2 \|f\| + \log_2 \|g\| + (d_f + d_g) \log_2 A. \quad (3)$$

Of course, the value taken by A will also depend on the size of the norms. If this size is larger than expected, then the probability of finding a relation is too small and the sieving range corresponding to A will not allow the creation of enough relations. But assuming A is fixed is enough to compare various types of polynomial constructions: if one of them gives larger norms, then for this construction, the value of A should be larger, leading to even larger norms. In other words, the optimal value is unchanged whether we consider A fixed or let it depend on d_f and d_g .

The best asymmetric construction is the classical SNFS. We first analyze the case where d_f and d_g are distinct. Let us assume $d_f > d_g$. We first remark that subject to constraint (2), Expression (3) is minimized by taking $\|f\|$ as small as possible (i.e. $\log_2 \|f\| = 0$) and $\log_2 \|g\| = n/d_f$. This yields an optimal norm size equal to $n/d_f + (d_f + d_g) \log_2 A$. It follows that given d_f , we should choose d_g to be minimal, which leads us precisely to the classical case, the example construction listed in the third row of Table 1. The optimal d_f yields an optimal norm size equal to $2\sqrt{n \log_2 A}$.

An all-balanced construction. In many situations, the optimal value is obtained by balancing each quantity as much as possible. Unfortunately, this is suboptimal in our case. If $d_f = d_g$, Expression (3) becomes $n/d_f + 2d_f \log_2 A$. Choosing the best possible value for d_f , we obtain $2\sqrt{2n \log_2 A}$. This is much worse than in the classical construction and in fact, pushing the analysis to its end would lead to a GNFS complexity with a $(64/9)^{1/3}$ exponent.

More general constructions. Unfortunately, it seems to be impossible to combine the SNFS construction with Coppersmith's multiple number field strategy [12,35,11] and obtain a complexity with an exponent smaller than $(32/9)^{1/3}$. Any linear combination of f and g will lead to a polynomial having both high degree and large coefficients, which must be avoided to achieve SNFS complexity.

In principle, one could also perform an analysis allowing skewed polynomials, where the ratio between two consecutive coefficients is roughly a constant different from 1. This general analysis would require still more parameters than the one we did, so we skip the details, since we did not find a situation where this could lead to a good asymptotic complexity.

4.2 Hiding the special form

The conclusion of the previous discussion is that the best form for a pair of SNFS polynomials is still the same as the one considered by Gordon more than 20 years ago. His discussion about how to hide them is still valid. We recall it here for completeness.

The goal is to find a prime p , and possibly a factor q of $p - 1$, together with an SNFS pair of polynomials, such that from the knowledge of p (and

q) it is harder to guess the SNFS pair of polynomials than to run a discrete logarithm computation with the general NFS algorithm, or using Pollard Rho in the subgroup of order q .

We enumerate requirements on the construction below.

The polynomial f must be chosen within a large enough set. If f is known, then its roots modulo p can be computed. With the Extended Euclidean Algorithm, it can be efficiently checked whether one of them is equal to a small rational number modulo p . If this is the case, then the numerator and the denominator are (up to sign) the coefficients of g . Therefore, if f has been chosen among a small set, an exhaustive search over the roots modulo p of all these polynomials will reveal the hidden SNFS pair of polynomials. Thus we must choose f from a large enough set so that this exhaustive search takes at least as much time as a direct discrete logarithm computation.

The two coefficients of g must be large. If g is a monic polynomial $g = x - g_0$, then, since $p = f(g_0)$, the most significant bits of p depend only on g_0 and the leading coefficient f_d of f . In that case, recovering the hidden SNFS polynomials reduces to an exhaustive search on the leading coefficient of f : we can use the LLL algorithm to minimize the other coefficients of f by writing a multiple of p as a sum of powers of g_0 . Examining the least significant bits of p shows that having a polynomial g with a constant term equal to 1 is equally bad. More generally, having one of the coefficients of g belonging to a small set also leads to a faster exhaustive search than if both are large. In the following, we will therefore always consider linear polynomials g for which the two coefficients have similar sizes; compared to using a monic g , this has only a marginal impact on the effectiveness of the SNFS efficiency in our context.

Attempts to unveil the trapdoor Heidi does not want her trapdoor to be unveiled, as she would not be able to plausibly deny wrongdoing. It is therefore highly important that Heidi convinces herself that the criteria above are sufficient for the trapdoor to be well hidden. We tried to improve on the method mentioned above that adapts to monic g . In particular, we tried to take advantage of the possibility that the leading coefficient of f might be divisible by small primes. This did not lead to a better method.

4.3 Adapting the prime to the hider's needs

Algorithm to build a DSA-like prime. In Algorithm 1, we recall the method of Gordon to construct hidden SNFS parameters in a DSA setting. The general idea is to start from the polynomial f and the prime q , then derive a polynomial g such that q divides the resultant of f and g minus 1, and only at the end check if this resultant is a prime p . This avoids the costly factoring of $p - 1$ that would be needed to check whether there is a factor of appropriate size to play

the role of q . Our version is slightly more general than Gordon's, since we allow signed coefficients for the polynomials. As a consequence, we do not ensure the sign of the resultant, so that the condition $q \mid p - 1$ can fail. This explains the additional check in Step 8. The size of the coefficients of f are also adjusted so that an exhaustive search on all the polynomials will take more or less the same time as the Pollard Rho algorithm in the subgroup of order q , namely $2^{s_q/2}$ where s_q is the bit-length of q .

In Step 6, it is implicitly assumed that 2^{s_q} is smaller than $2^{s_p/d}/\|f\|$, that is $s_q < s_p/d - s_q/2(d+1)$. This condition will be further discussed in the next subsection. We note however that if it fails to hold by only a few bits, it is possible to run the algorithm and hope that the root r produced at Step 5 will be small enough. We can expect that r will behave like a uniformly random element modulo q , so that the probability that this event occurs can be estimated.

Input : The bit-sizes s_p and s_q for p and q ; the degree d of f .
Output: HSNFS parameters f, g, p, q .

- 1 Pick a random irreducible polynomial f , with $\|f\| \approx 2^{s_q/2(d+1)}$;
- 2 Pick a random prime q of s_q bits;
- 3 Pick a random integer $g_0 \approx 2^{s_p/d}/\|f\|$;
- 4 Consider the polynomial $G_1(g_1) = \text{Res}_x(f(x), g_1x + g_0) - 1$ of degree d in g_1 ;
- 5 Pick a root r of G_1 modulo q ; if none exists go back to Step 1;
- 6 Add a random multiple of q to r to get an integer g_1 of size $\approx 2^{s_p/d}/\|f\|$;
- 7 Let $p = |\text{Res}_x(f(x), g_1x + g_0)|$;
- 8 If p has not exactly s_p bits or if p is not prime or if q does not divide $p - 1$, then go back to Step 1;
- 9 Return f, g, p, q .

Algorithm 1: Gordon's hidden SNFS construction algorithm

Selecting good f -polynomials. In Algorithm 1, in the case of failure at Step 5 or Step 8, we could restart only at Step 2, in order to keep using the same f -polynomial for a while. More generally, the polynomial f could be given as input of the algorithm, opening the opportunity for the hider to use a polynomial f with nice algebraic properties that accelerate the NFS algorithm. The so-called Murphy- α value [37, §3.2] has a measurable influence on the probability of the norms to be smooth. A norm of s bits is expected to have a smoothness probability similar to the one of a random integer of $s + \frac{\alpha}{\log 2}$ bits. A negative α -value is therefore helping the relation collection.

Experimentally, for an irreducible polynomial of fixed degree over \mathbb{Z} with coefficients uniformly distributed in an interval, the α -value follows a centered normal law with standard deviation around 0.94 (measured empirically for degree 6). From this, it is possible to estimate the expected minimum α -value after trying N polynomials: we get $\alpha_{\min} \sim -0.94\sqrt{2\log N}$.

In a set of 2^{80} candidates for the f -polynomial, we can therefore expect to find one with an α -value around -10 . But it is *a priori* very hard to find this polynomial, and if it were easy, then it would not be a good idea for the hider to choose it, because then it would not be hidden anymore. A compromise is for the hider to try a small proportion of the candidates and keep the one with the best α . Since checking the α -value of a polynomial is not really faster than checking its roots modulo p , the attacker gains no advantage by knowing that f has a smaller value than average. For instance, after trying 2^{20} polynomials, one can expect to find an f that has an α -value of -5 which gives a nice speed-up for the NFS without compromising the hidden property.

Apart from the α -value, another well-known feature of polynomials that influences the smoothness properties is the number of real roots: more real roots translates into finding relations more easily. We did not take this into account in our proof of concept experiments, but this could certainly also be used as a criterion to select f .

4.4 Size considerations

Algorithm 1 does not work if the size s_q of the subgroup order q is too large compared to the size of the coefficients of the g -polynomials that are optimal for the size of p . The condition is

$$s_q < s_p/d - s_q/2(d+1),$$

where d is the optimal degree of the f -polynomial for running SNFS on a prime of s_p bits. We can plug in the asymptotic formula for d in terms of s_p : it is proportional to $(s_p/\log(s_p))^{1/3}$, leading to a condition of the form

$$s_q < c(s_p)^{2/3}(\log(s_p))^{1/3} = \log(L_p(2/3, c)),$$

for a constant c . Now, s_q will be chosen so that the running time of Pollard Rho in the subgroup of order q matches the running time of the NFS algorithm modulo p . The former grows like $2^{s_q/2}$, while the latter grows like $L_p(1/3, c') \approx 2^{s_p^{1/3}}$. Therefore, asymptotically, it makes sense to have s_q close to proportional to $s_p^{1/3}$, and the condition for Algorithm 1 to work is easily satisfied.

Back in 1992, when Gordon studied the question, the complexity analysis of the Number Field Sieve was not as well understood, and the available computing power was far less than today. At that time, $s_q = 160$ and $s_p = 512$ were the proposed parameter sizes for DSA, leading to difficulties satisfying the condition of Algorithm 1 unless a suboptimal d was chosen. Nowadays, popular DSA parameters are $s_p = 1024$ and $s_q = 160$, leaving much room for the condition to hold, and it is possible to choose $d = 6$, which is optimal for our NFS implementation. Therefore, the relevant parameters for today are beyond the point where Gordon's algorithm would need to be run with suboptimal parameters.

4.5 Reassessing the hiding problem

The prudent conclusions of cryptographers in the 1990’s was that it might be difficult to put a useful and hard to detect trapdoor in a DSA prime. For example in [15], Lenstra concludes that “this kind of trap can be detected”, based on the trap design from [31]. It is true that whether for Lenstra’s trap method in [31], or Gordon’s trap in Algorithm 1, f had to be chosen within a too-small set given the state of the art with NFS back in 1992. This stance is also found in reference books from the time, such as the *Handbook of Applied Cryptography* by Menezes, van Oorschot, and Vanstone [36, note §8.4] which remain influential today.

This is no longer true. It is now clearly possible to hide an SNFS pair of polynomials for a DSA prime p of 1024 bits with a 160-bit subgroup. It remains to show that this SNFS computation is indeed feasible, even with moderate academic computing resources.

5 Computation of a 1024-bit SNFS DLP

In addition to the computational details, we describe the algorithmic improvements and parameter choices that played a key role in the computation.

	sieving	linear algebra			individual log
		sequence	generator	solution	
cores	≈3000	2056	576	2056	500–352
CPU time (1 core)	240 years	<u>123 years</u>	<u>13 years</u>	<u>9 years</u>	10 days
calendar time	1 month	1 month			80 minutes

Table 2. Our 1024-bit hidden SNFS discrete log computation took around two months of calendar time to complete. We used a variety of resources for sieving, so the total number of cores in use varied over time.

5.1 Selecting a target

We ran Algorithm 1 to find a hidden SNFS prime p of 1024 bits such that \mathbb{F}_p has a subgroup of prime order q of 160 bits. For these parameters, a polynomial f of degree $d = 6$ is the most appropriate. After a small search among the polynomials with (signed) coefficients of up to 11 bits, we selected

$$f = 1155x^6 + 1090x^5 + 440x^4 + 531x^3 - 348x^2 - 223x - 1385,$$

for which the α -value is about -5.0 . The set of all polynomials with this degree that satisfy the coefficient bound is a bit larger than 2^{80} , which is the expected cost of Pollard Rho modulo q . We note that the cost of testing a polynomial f

(a root finding modulo p and the rational reconstruction of these roots) is much higher than one step of Pollard Rho (one multiplication modulo p), so this is a conservative setting.

We then ran the rest of Algorithm 1 exactly as it is described. The resulting public parameters are

$$\begin{aligned}
 p = & 163323987240443679101402070093049155030989439806917519173580070791569 \\
 & 227728932850358498862854399351423733697660534800194492724828721314980 \\
 & 248259450358792069235991826588944200440687094136669506349093691768902 \\
 & 440555341493237296555254247379422702221515929837629813600812082006124 \\
 & 038089463610239236157651252180491 \\
 q = & 1120320311183071261988433674300182306029096710473 ,
 \end{aligned}$$

and the trapdoor polynomial pair is

$$\begin{aligned}
 f &= 1155x^6 + 1090x^5 + 440x^4 + 531x^3 - 348x^2 - 223x - 1385 \\
 g &= 567162312818120432489991568785626986771201829237408x \\
 &\quad - 663612177378148694314176730818181556491705934826717 .
 \end{aligned}$$

This computation took 12 core-hours, mostly spent in selecting a polynomial f with a good α -value. No effort was made to optimize this step.

5.2 Choosing parameters for the sieving step

The sieving step (also known as relation collection) consists of finding many (a, b) -pairs such that the two norms $\text{Res}(f(x), a - bx)$ and $\text{Res}(g(x), a - bx)$ are simultaneously smooth.

We use the special- q sieving strategy, where we concentrate the search in the positions where we know in advance that one of the two norms will be divisible by a large prime: the special q . For the general number field sieve, it is always the case that one norm is much larger than the other, so it makes sense to choose the special q on the corresponding side. In our case, the norms have almost the same size (about 200 bits each), so there is no obvious choice. Therefore, we decided to sieve with special q 's on both sides. As a consequence, the largest special q that we had to consider were 1 or 2 bits smaller than if we had allowed special q 's to be only on one side; the norms were accordingly a bit smaller.

The general strategy used for a given special q is classical: among a vast quantity of candidates, we mark those that are divisible by primes up to a given *sieving bound* using a sieve *à la* Eratosthenes; then the most promising candidates are further scrutinized using the ECM method, trying to extract primes up to the smoothness bound B . The criterion for selecting those promising candidates is best expressed as the number of times the smoothness bound is allowed for the remaining part of the norms once the sieved primes have been removed. This is usually referred to as the *number of large primes* allowed on a given side.

For the 1024-bit computation, we used the following parameters. On the rational side, we sieved all the prime special q in the 150M–1.50G range (that is, with $1.5 \cdot 10^8 < q < 1.5 \cdot 10^9$), and on the algebraic side, we sieved special- q

prime ideals in the range 150M–1.56G. The difference between the two is due to the fact that we used two different clusters for this step, and when we stopped sieving, one was slightly ahead of the other.

For each special q , the number of (a, b) -pairs considered was about 2^{31} . This number includes the pairs where both a and b are even, but almost no time is spent on those, since they cannot yield a valid relation.

All primes on both sides up to 150M were extracted using sieving, and the remaining primes up to the smoothness bound $B = 2^{31}$ were extracted using ECM. On the side where the special q was placed, we allowed 2 large primes, while 3 large primes were allowed on the other side.

This relation collection step can be parallelized almost infinitely with no overhead since each special q is handled completely separately from the others. We used a variety of computational resources for the sieving, and in general took advantage of hyperthreading and in addition oversubscribed our virtual cores with multiple threads. Aggregating reported CPU time for virtual cores over all of the machine types we used, we spent $5.08 \cdot 10^9$ CPU seconds, or 161 CPU years sieving the rational side, and $5.03 \cdot 10^9$ CPU seconds, or 159 CPU years sieving the algebraic side. In order to obtain a more systematic estimate of the CPU effort dedicated to sieving without these confounding factors, we ran sampling experiments on a machine with 2 Intel Xeon E5-2650 processors running at 2.00 Ghz with 16 physical cores in total. From these samples, we estimate that sieving would have taken 15 years on this machine, or 240 core-years. We spent about one month of calendar time on sieving.

The total number of collected relations was 520M relations: 274M from the rational side and 246M from the algebraic side. Among them, 249M were unique, involving 201M distinct prime ideals. After filtering these relations, we obtained a matrix with 28M rows and columns, with 200 non-zero entries per row on average.

Before entering the linear algebra step, we calculated the dense block of “Schirokauer maps”, which are q -adic characters introduced by Schirokauer in [46]. These consist, for each matrix row, in 3 full-size integers modulo q (the number 3 is here the unit rank of the number field defined by our polynomial f).

5.3 Linear algebra

The linear algebra problem to be solved can be viewed in several ways. One is to consider a square matrix of size $N \times N$, whose left-hand side M of size $N \times (N-r)$ is the matrix produced by the filtering task, while the right block S of size $N \times r$ is made of dense Schirokauer maps. Recent work [24] has coined the term “nearly sparse” for such matrices. We seek a non-trivial element of the right nullspace of the square matrix $(M||S)$.⁵ This approach has the drawback that an iterative

⁵ The integer factorization case, in contrast, has $q = 2$, and requires an element of the *left* nullspace. The latter fact allows for a two-stage algorithm selecting first many solutions to $xM = 0$, which can then be recombined to satisfy $x(M||S) = 0$. No such approach works for the right nullspace.

linear algebra algorithm based on the matrix $(M\|S)$ is hampered by the weight of the block S , which contributes to each matrix-times-vector product.

Shirokauer maps serve as initialization vectors An alternative method, originally proposed by Coppersmith [13, §8] alongside the introduction of the block Wiedemann algorithm, is to use this algorithm to constructively write a zero element in the sum of the column spaces of M and S . In this case, the iterative algorithm is run on the square matrix $M_0 = (M\|0)$, which is of considerably lower weight than $(M\|S)$. More precisely, the block Wiedemann algorithm, with two *blocking factors* which are integers m and n , achieves this by proceeding through the following steps. The blocking factor n is chosen so that $n \geq r$, and we let $D(t)$ be the diagonal $n \times n$ matrix with coefficients t (r times) and 1 ($n - r$ times).

Initialization. Pick blocks of *projection vectors* $x \in \mathbb{F}_q^{N \times m}$ and *starting vectors* $y \in \mathbb{F}_q^{N \times n}$. The block x is typically chosen of very low weight, while we set $y = (S\|R)$, with R a random block in $\mathbb{F}_q^{N \times (n-r)}$.

Sequence. Compute the sequence of matrices $a_i = {}^t x M_0^i y$, for $0 \leq i < L$, with $L = \lceil N/m \rceil + \lceil N/n \rceil + \lceil m/n + n/m \rceil$.

Linear generator. Let $A(t) = \sum_i a_i t^i$. Let $A'(t) = A(t)D(t) \operatorname{div} t$. Compute an $n \times n$ matrix of polynomials $F(t)$ such that $A'(t)F(t)$ is a matrix of polynomials of degree less than $\deg F$, plus terms of degree above L (see [13,52]). We typically have $\deg F \approx N/n$.

Solution. Consider one column of degree d of $F(t)$. Write the corresponding column of $D(t)F(t)$ as $ct^{d+1} + f_0 t^d + \dots + f_d$ with $c, f_i \in \mathbb{F}_q^{n \times 1}$. With high probability, we have $c \neq 0$ and $w = (S\|0)c + M_0 \sum_{i \geq 0} M_0^i y f_i = 0$. Rewrite that as $Mu + Sv = 0$, where u and v are:

$$u = \text{first } N - r \text{ coefficients of } \sum_{i \geq 0} M_0^i y f_i,$$

$$v = \text{first } r \text{ coefficients of } c$$

This readily provides a solution to the problem.

Solving the linear system with $(1 + o(1))N$ SpMVs The most expensive steps above are the *sequence* and *solution* steps. The dominating operation is the sparse matrix-times-vector operation (SpMV), which multiplies M_0 by a column vector in $\mathbb{F}_q^{N \times 1}$. It is easy to see that the sequence step can be run as n independent computations, each requiring L SpMV operations (therefore $nL = (1 + n/m)N$ in total): matrices a_i are computed piecewise, column by column. Once all these computations are completed, the fragments of the matrices a_i need to be collated to a single place in order to run the linear generator step.

It is tempting to regard the solution step in a directly similar way (as was done, e.g., in the context of the recent 768-bit DLP computation [28]). However, as was pointed out very early on by Kaltofen [26, Step C3, p. 785, and corollary

to Theorem 7] yet seems to have been overlooked since, one should proceed differently. Assume that some of the vectors $M_0^i y$ from the sequence step have been kept as regular checkpoints (an obvious choice is $M_0^{Kj} y$ for some well chosen checkpoint period K). For an arbitrary j , we compute $\sum_{i=0}^{i=K-1} M_0^i M_0^{Kj} y f_{Kj+i}$ with a Horner evaluation scheme which costs K SpMV operations only. These expressions together add up to u , and can be computed independently (using as many independent tasks as K allows). This adds up to $\deg F \approx N/n$ SpMV operations.

In total, this evaluation strategy yields a cost of $(1 + n/m + 1/n)N$ SpMV operations (see [26, Theorem 7]), which can be freely parallelized n -fold for the sequence step, and possibly much more for the solution step. It is important to note that as blocking factors m and n grow with $m \gg n$, this brings the total cost close to N SpMV operations, a count which to our knowledge beats all other exact sparse linear algebra algorithms. The only limiting point to that is the linear generator step, whose cost depends roughly linearly on $(m + n)$. Thanks to the use of asymptotically fast algorithms [7,52,20], this step takes comparatively little time.

Linear algebra for 1024-bit SNFS DLP The matrix M_0 had 28 million rows and columns, and 200 non-zero coefficients per row on average. We used the linear algebra code in CADO-NFS [51]. We chose blocking factors $m = 24$, $n = 12$. Consequently, a total of 44 million SpMV operations were needed. We ran these in two computing facilities in the respective research labs of the authors, with a roughly even split. For the sequence step, each of the 12 independent computations used between 4 and 8 nodes, each with up to 44 physical cores. The nodes we used were interconnected with various fabrics, including Mellanox 56 Gbps Infiniband FDR and 40 Gbps Cisco UCS Interconnects. The total time for the sequence step was about 123 core-years. The linear generator step was run on 36 nodes, and cost 13 core-years. The solution step was split in $48 = \frac{\deg F = 2400000}{K = 50000}$ independent tasks. Each used a fairly small number of nodes (typically one or two), which allowed us to minimize the communication cost induced by the parallelization. Despite this, each iteration was 33% more expensive than the ones for the sequence step, because of the extra cost of the term $M_0^{Kj} y f_{Kj+i}$ which is to be added at each step. The total time for the solution step was 9 core-years, which brings the total linear algebra cost for this computation below 150 core-years. In total we spent about one month of calendar time on linear algebra. Table 4 in Appendix B gives more details of the iteration times for the different machine architectures present in our clusters.

After this step and propagating the knowledge to relations that were eliminated during the filtering step we obtained the logarithm of 198M elements, or 94.5% of the prime ideals less than 2^{31} .

5.4 Individual logarithms

In our scenario where a malicious party has generated a trapdoored prime with a goal of breaking many discrete logarithms in the corresponding group, it is

interesting to give details on the individual logarithm step, which is often just quickly mentioned as an “easy” step (with the notable exception of [2] where it is at the heart of the attack).

From now on, we denote by z the element of the group G for which we want the discrete logarithm (modulo q). The database of discrete logarithms computed thus far is with respect to an a priori unknown generator. In order to obtain the logarithm of z with respect to a generator specified by the protocol being attacked, it is typical that two individual logarithm queries are necessary. This aspect will not be discussed further.

The individual logarithm step can itself be decomposed in two sub-steps:

Initialization Find an exponent e such that $z' = z^e \equiv u/v \pmod{p}$, where u and v are B_{init} -smooth numbers of size about half of the size of p . Note that B_{init} has to be much larger than B to get a reasonable smoothness probability.

Descent For every factor of u or v that is larger than the smoothness bound, treat it as a special- q to rewrite its discrete logarithm in terms of smaller elements, and continue recursively until it is rewritten in terms of elements of known logarithm.

We emphasize that the Initialization step does not use the polynomials selected for the NFS computation, and therefore, it does not take advantage of the SNFS nature of the prime p . For the Descent step, on the other hand, this makes heavy use of the polynomials, and here knowing the hidden polynomials corresponding to p helps significantly.

Asymptotic complexities In terms of asymptotic complexity, the Initialization step is more costly than the Descent step, and in a theoretical analysis, the bound B_{init} is chosen in order to minimize the expected time of the Initialization step only. The early-abort analysis of Barbulescu [5, Chapter 4] gives $B_i = L_p(2/3, 0.811)$, for a running time of $L_p(1/3, 1.232)$.

For the Descent step, the complexity analysis can be found in two different flavours in the literature: either we use polynomials of degree 1 (the polynomial $a - bx$ corresponding to an (a, b) -pair) like in [48], or polynomials of possibly higher degrees, depending on where we are in the descent tree similarly to [17].

Using higher degree polynomials, we get a complexity of $L_p(1/3, 0.763)$, where the last steps of the descent are the most costly. Sticking with polynomials of degree 1, the first steps become more difficult and the complexity is $L_p(1/3, 1.117)$. Both are lower than the complexity of the Initialization step.

We give further details on the complexity analysis of individual logarithms in Appendix A.

Practical approach This theoretical behaviour gives only a vague indication of the situation for our practical setting. For the initialization step, we follow the general idea of Joux-Lercier [23]. For a random integer e , we compute $z' \equiv z^e \pmod{p}$, and take two consecutive values in the extended Euclidean algorithm to

compute u_0, v_0, u_1, v_1 of size about \sqrt{p} such that $z' \equiv \frac{u_0}{v_0} \equiv \frac{u_1}{v_1} \pmod{p}$. We then look for two integers a and b such that $u = au_0 + bu_1$ and $v = av_0 + bv_1$ are both smooth. Since this is an unlikely event, and testing for smoothness is very costly, we do a 3-step filtering strategy.

First, we sieve on the set of small (a, b) -pairs, to detect pairs for which the corresponding u and v are divisible by many small primes. For this, we re-use the sieving code that helps collecting relations. After this first step, we keep only (a, b) -pairs for which the remaining unfactored part is less than a given threshold on each side.

Then, many ECM factorizations are run on each remaining cofactor; this is tuned so that we expect most of the prime factors up to a given bit size to be extracted. After this step, we again keep only the candidates for which the remaining unfactored parts are smaller than another threshold. The cofactors of the surviving pairs are then fully factored using MPQS.

At each stage, if a prime factor larger than the smoothness bound B is found, we naturally abort the computation.

This practical strategy keeps the general spirit of filters used in the theoretical analysis of Barbulescu which relies only on ECM, but we found that combined with sieving and MPQS, it is much faster.

Parameters for the 1024-bit SNFS computation For the 1024-bit computation, we used a bound $B_{\text{init}} = 135$ bits for this Initialization step. After applying the Joux-Lercier trick, we first sieved to extract primes up to 2^{31} , and we kept candidates for which the unfactored parts are both less than 365 bits. Then we used GMP-ECM with 600 curves and $B_1 = 500,000$, hoping to remove most of the prime factors of 100 bits and less. After this second step, we kept only the candidates for which the unfactored parts are both less than 260 bits.

For the Descent step, we used only polynomials of degree 1. Polynomials of degree 2 do not seem to yield smaller norms even for the largest 135-bit primes to be descended. The depth of the recursion was rarely more than 7. A typical example of a sequence of degrees encountered while following the tree from the top to a leaf is

$$135 \rightarrow 90 \rightarrow 65 \rightarrow 42 \rightarrow 33 \rightarrow 31,$$

but this is of course different if the ideals along the path are on the rational or the algebraic side.

Both the Initialization and the Descent steps can be heavily parallelized. The expected CPU-time for computing an individual logarithm is a few days on a typical core, distributed more or less equally between the two steps. Using parallelization, we managed to get a result in 80 minutes of wall-clock time: the initialization took around 20 minutes parallelized across 500 cores, and the descent took 60 minutes parallelized across 352 cores. (We used 44 cores for each large special q to be descended.)

As an example, we computed the discrete logarithm of

$$z = \lceil \pi 10^{307} \rceil = 3141592653 \dots 7245871,$$

taking 2 as a generator. More precisely, we are talking about their images in the subfield of order q obtained by raising 2 and z to the power $(p-1)/q$. We obtained:

$$\log z / \log 2 \equiv 409559101360774669359443808489645704082239513256 \pmod{q},$$

which can be easily checked to be the correct answer.

6 Discussion

6.1 Comparison with GNFS DLP for various sizes

The recently reported GNFS 768-bit discrete logarithm computation [28] took about 5000 core-years. It is tempting to directly compare this number to the 400 core-years that we spent in our experiments. As a rule of thumb, one would expect the 768-bit GNFS to be about a factor of 10 more difficult than a 1024-bit SNFS computation, and this appears to hold in the numbers we report. However, we note that first, the software used in both experiments are different (the CADO-NFS sieving implementation is slower than Kleinjung's), and second, the GNFS-768 computation was done with a safe prime, while we used a DSA prime, thus saving a factor of 6 in the linear algebra running time.

It is possible to get another hint for the comparison by considering the typical sizes of the norms in both contexts. For GNFS-768, they appear to be roughly 20 bits larger (in total) than for SNFS-1024. Taking all correcting factors into account, like the α -values, the (un-)balance of the norms, and the special- q , this translates to roughly a factor of 8 in the smoothness probability, thus more or less confirming the ratio of running times observed in practice.

Asymptotically, the difference of complexities between GNFS and SNFS (namely the factor $2^{1/3}$ in the exponent) means that we would expect to obtain similar running times when SNFS is run on an input that is twice the size of the one given to GNFS. However, key sizes relevant to current practice and practical experiments are still too small for these asymptotic bounds to be accurate.

To get concrete estimates for these smaller key sizes, we can compare the size of the norms and estimate that an 1152-bit SNFS computation would correspond to the same amount of time as a GNFS-768. For an SNFS of 2048 bits, the equivalent would be around a GNFS of 1340 bits. And finally, for an SNFS of 4096 bits, the equivalent would be around a GNFS of 2500 bits. Of course, for such large sizes these are more educated guesses than precise estimates.

6.2 Non-hidden SNFS primes in real use

We have found multiple implementations using non-hidden SNFS primes in the real world. 150 hosts used the 512-bit prime $2^{512} - 38117$ for export-grade Diffie-Hellman key exchange in a full IPv4 HTTPS scan performed by Adrian et al. [2]

in March 2015. Performing the full NFS discrete log computation for this prime took about 215 minutes on 1288 cores, with 8 minutes spent on the sieving stage, 145 minutes spent on linear algebra, and the remaining time spent filtering relations and reconstructing logarithms. In September 2016, 134 hosts were observed still using this prime.

We also found 170 hosts using the 1024-bit prime $2^{1024} - 1093337$ for non-export TLS Diffie-Hellman key exchange in scans performed by Adrian et al. In September 2016, 106 hosts were still using this prime. We estimate that performing a SNFS-DL computation for this prime would require about 3 times the amount of effort for the sieving step as the 1024-bit SNFS computation that we performed. This difference is mostly due to the α -value of the f -polynomial that can not easily be made small. The linear algebra step will suffer at the very least a 7-fold slowdown. Indeed, since this prime is safe, the linear algebra must be performed modulo $(p - 1)/2$, which is more expensive than the 160-bit linear algebra we used for a DSA prime in our computation. Furthermore, since the smoothness probabilities are worse, we expect also the matrix to be a bit larger, and the linear algebra step cost to grow accordingly.

The LibTomCrypt library [14], which is widely distributed and provides public domain implementations of a number of cryptographic algorithms, includes several hard-coded choices for Diffie-Hellman groups ranging in size from 768 to 4096 bits. Each of the primes has a special form amenable to the SNFS. The 768-bit strength (actually a 784-bit prime) is $2^{784} - 2^{28} + 1027679$. We performed a SNFS discrete log for this prime. On around a thousand cores, we spent 10 calendar days sieving and 13 calendar days on linear algebra. The justification for the special-form primes appears to be the diminished radix form suggested by Lim and Lee [34], which they suggest for decreasing the cost of modular reduction. We examined the TLS and SSH scan datasets collected by Adrian et al. [2] and did not find these primes in use for either protocol.

We also carried out a perfunctory search for poorly hidden SNFS primes among public key datasets, based on the rather straightforward strategy in §4.2, hoping for monic g , and f such that $2 \leq d \leq 9$ and $|f_d| \leq 1024$. We carried out this search for the 11,658 distinct SSH group exchange primes, 68,126 distinct TLS ephemeral Diffie-Hellman primes, and 2,038,232 distinct El Gamal and DSA primes from a dump of the PGP public key database. This search rediscovered the special-form TLS primes described above, but did not find any other poorly hidden primes susceptible to SNFS. We cannot rule out the existence of trapdoored primes using this method, but if hidden SNFS primes are in use the designers must have followed Gordon’s advice.

6.3 Lessons

It is well known among the cryptographic community that 1024-bit primes are insufficient for cryptosystems based on the hardness of discrete logarithms. Such primes should have been removed from use years ago. NIST recommended transitioning away from 1024-bit key sizes for DSA, RSA, and Diffie-Hellman in

2010 [6]. Unfortunately, such key sizes remain in wide use in practice. Our results are yet another reminder of the risk, and we show this dramatically in the case of primes which lack verifiable randomness. The discrete logarithm computation for our backdoored prime was only feasible because of the 1024-bit size.

The asymptotic running time estimates suggest that a SNFS-based trapdoor for a 2048-bit key would be roughly equivalent to a GNFS computation for a 1340-bit key. We estimate that such a computation is about 16 million times harder than the 1024-bit computation that we performed, or about $6.4 \cdot 10^9$ core-years. Such a computation is likely still out of range of even the most sophisticated adversaries in the near future, but is well below the security guarantees that a 2048-bit key should provide. Since 2048-bit keys are likely to remain in wide usage for many years, standardized primes should be published together with their seeds.

In the 1990s, key sizes of interest were largely limited to 512 or 1024 bits, for which a SNFS computation was already known to be feasible in the near future. Both from this perspective, and from our more modern one, dismissing the risk of trapdoored primes in real usage appears to have been a mistake, as the apparent difficulties encountered by the trapdoor designer in 1992 turn out to be easily circumvented. A more conservative design decision for FIPS 186 would have required mandatory seed publication instead of making it optional. As a result, there are opaque, standardized 1024-bit and 2048-bit primes in wide use today that cannot be properly verified.

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References

1. — (author redacted). Eurocrypt '92 reviewed. Cryptolog, March 1994. <https://www.nsa.gov/news-features/declassified-documents/cryptologs/>.

2. D. Adrian, K. Bhargavan, Z. Durumeric, P. Gaudry, M. Green, J. A. Halderman, N. Heninger, D. Springall, E. Thomé, L. Valenta, B. VanderSloot, E. Wustrow, S. Z. Béguelin, and P. Zimmermann. Imperfect forward secrecy: How Diffie-Hellman fails in practice. In I. Ray, N. Li, and C. Kruegel, editors, *ACM CCS 15: 22nd Conference on Computer and Communications Security*, pages 5–17, Denver, CO, USA, Oct. 12–16, 2015. ACM Press.
3. K. Aoki, J. Franke, T. Kleinjung, A. K. Lenstra, and D. A. Osvik. A kilobit Special Number Field Sieve factorization. In K. Kurosawa, editor, *Advances in Cryptology – ASIACRYPT 2007*, pages 1–12. Springer Berlin Heidelberg, 2007.
4. J. Ball, J. Borger, and G. Greenwald. Revealed: how US and UK spy agencies defeat internet privacy and security. *The Guardian*, Sep. 5 2013. Online: <https://www.theguardian.com/world/2013/sep/05/nsa-gchq-encryption-codes-security>.
5. R. Barbulescu. *Algorithmes de logarithmes discrets dans les corps finis*. PhD thesis, Université de Lorraine, France, 2013.
6. E. Barker and A. Roginsky. Transitions: Recommendation for transitioning the use of cryptographic algorithms and key lengths. Technical report, National Institute of Standards and Technology, 2011. <http://nvlpubs.nist.gov/nistpubs/Legacy/SP/nistspecialpublication800-131a.pdf>.
7. B. Beckerman and G. Labahn. A uniform approach for the fast computation of matrix-type Padé approximants. *SIAM J. Matrix Anal. Appl.*, 15(3):804–823, Jul. 1994.
8. D. J. Bernstein, T. Chou, C. Chuengsatiansup, A. Hülsing, E. Lambooi, T. Lange, R. Niederhagen, and C. van Vredendaal. How to manipulate curve standards: A white paper for the black hat. In L. Chen and S. Matsuo, editors, *Security Standardisation Research - Second International Conference, SSR 2015*, volume 9497 of *Lecture Notes in Comput. Sci.*, pages 109–139. Springer, 2015.
9. S. Checkoway, M. Fredrikson, R. Niederhagen, A. Everspaugh, M. Green, T. Lange, T. Ristenpart, D. J. Bernstein, J. Maskiewicz, and H. Shacham. On the practical exploitability of Dual EC in TLS implementations. In K. Fu, editor, *Proceedings of USENIX Security 2014*, pages 319–35. USENIX, Aug. 2014.
10. S. Checkoway, J. Maskiewicz, C. Garman, J. Fried, S. Cohney, M. Green, N. Heninger, R.-P. Weinmann, E. Rescorla, and H. Shacham. A systematic analysis of the juniper dual EC incident. In E. R. Weippl, S. Katzenbeisser, C. Kruegel, A. C. Myers, and S. Halevi, editors, *ACM CCS 16: 23rd Conference on Computer and Communications Security*, pages 468–479, Vienna, Austria, Oct. 24–28, 2016. ACM Press.
11. A. Commeine and I. Semaev. An algorithm to solve the discrete logarithm problem with the number field sieve. In M. Yung, Y. Dodis, A. Kiayias, and T. Malkin, editors, *PKC 2006*, volume 3958 of *Lecture Notes in Comput. Sci.*, pages 174–190. Springer, Heidelberg, 2006.
12. D. Coppersmith. Modifications to the number field sieve. *Journal of Cryptology*, 6(3):169–180, 1993.
13. D. Coppersmith. Solving linear equations over $GF(2)$ via block Wiedemann algorithm. *Math. Comp.*, 62(205):333–350, Jan. 1994.
14. T. S. Denis. LibTomCrypt. <http://www.libtom.net/>.
15. Y. Desmedt, P. Landrock, A. K. Lenstra, K. S. McCurley, A. M. Odlyzko, R. A. Rueppel, and M. E. Smid. The Eurocrypt '92 controversial issue: Trapdoor primes and moduli (panel). In R. A. Rueppel, editor, *EUROCRYPT'92*, volume 658 of *Lecture Notes in Comput. Sci.*, pages 194–199. Springer, Heidelberg, May 1993.

16. Z. Durumeric, D. Adrian, A. Mirian, M. Bailey, and J. A. Halderman. A search engine backed by internet-wide scanning. In I. Ray, N. Li, and C. Kruegel, editors, *ACM CCS 15: 22nd Conference on Computer and Communications Security*, pages 542–553, Denver, CO, USA, Oct. 12–16, 2015. ACM Press.
17. A. Enge, P. Gaudry, and E. Thomé. An $L(1/3)$ discrete logarithm algorithm for low degree curves. *Journal of Cryptology*, 24(1):24–41, Jan. 2011.
18. M. Friedl, N. Provos, T. de Raadt, K. Steves, D. Miller, D. Tucker, J. McIntyre, T. Rice, and B. Lindstrom. Announce: OpenSSH 7.0 released, august 2015. <http://www.openssh.com/txt/release-7.0>.
19. D. K. Gillmor. Negotiated FFDHE for TLS, august 2016. <https://datatracker.ietf.org/doc/rfc7919/>.
20. P. Giorgi and R. Lebreton. Online order basis algorithm and its impact on the block Wiedemann algorithm. In *ISSAC '14*, pages 202–209. ACM, 2014.
21. D. M. Gordon. Designing and detecting trapdoors for discrete log cryptosystems. In E. F. Brickell, editor, *CRYPTO'92*, volume 740 of *Lecture Notes in Comput. Sci.*, pages 66–75. Springer, Heidelberg, Aug. 16–20, 1993.
22. D. M. Gordon. Discrete logarithms in $GF(p)$ using the number field sieve. *SIAM J. Discrete Math.*, 6(1):124–138, Feb. 1993.
23. A. Joux and R. Lercier. Improvements to the general number field sieve for discrete logarithms in prime fields. A comparison with the gaussian integer method. *Math. Comp.*, 72(242):953–967, Apr. 2003.
24. A. Joux and C. Pierrot. Nearly sparse linear algebra and application to discrete logarithms computations. In A. Canteaut, G. Effinger, S. Huczynska, D. Panario, and L. Storme, editors, *Contemporary Developments in Finite Fields and Applications*, pages 119–144. World Scientific Publishing Company, 2016.
25. Juniper Networks. 2015-12 Out of Cycle Security Bulletin: ScreenOS: Multiple Security issues with ScreenOS (CVE-2015-7755, CVE-2015-7756), Dec. 2015.
26. E. Kalfoten. Analysis of Coppersmith’s block Wiedemann algorithm for the parallel solution of sparse linear systems. *Math. Comp.*, 64(210):777–806, Apr. 1995.
27. T. Kleinjung, K. Aoki, J. Franke, A. K. Lenstra, E. Thomé, J. W. Bos, P. Gaudry, A. Kruppa, P. L. Montgomery, D. A. Osvik, H. J. J. te Riele, A. Timofeev, and P. Zimmermann. Factorization of a 768-bit RSA modulus. In T. Rabin, editor, *CRYPTO 2010*, volume 6223 of *Lecture Notes in Comput. Sci.*, pages 333–350. Springer, Heidelberg, Aug. 15–19, 2010.
28. T. Kleinjung, C. Diem, A. K. Lenstra, C. Priplata, and C. Stahlke. Discrete logarithms in $GF(p)$ — 768 bits. E-mail on the NMBRTHRY mailing list, June 16, 2016.
29. O. M. Kolkman, W. M. Mekking, and R. M. Gieben. DNSSEC Operational Practices, Version 2. RFC 6781, Internet Society, December 2012.
30. J. Larson, N. Perlroth, and S. Shane. Revealed: The NSA’s secret campaign to crack, undermine internet security. *ProPublica*, Sep. 5 2013. Online: <https://www.propublica.org/article/the-nsas-secret-campaign-to-crack-undermine-internet-encryption>.
31. A. K. Lenstra. Constructing trapdoor primes for the proposed DSS. Technical report, 1991. <https://infoscience.epfl.ch/record/164559>.
32. A. K. Lenstra and H. W. Lenstra, Jr., editors. *The development of the number field sieve*, volume 1554 of *Lecture Notes in Math.* Springer-Verlag, 1993.
33. M. Lepinski and S. Kent. Additional Diffie-Hellman groups for use with IETF standards, 2010. <http://ietf.org/rfc/rfc5114.txt>.
34. C. H. Lim and P. J. Lee. Generating efficient primes for discrete log cryptosystems, 2006. <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.43.8261>.

35. D. V. Matyukhin. On asymptotic complexity of computing discrete logarithms over $GF(p)$. *Discrete Mathematics and Applications*, 13(1):27–50, 2003.
36. A. J. Menezes, P. C. van Oorschot, and S. A. Vanstone. *Handbook of Applied Cryptography*. CRC Press, 1997.
37. B. A. Murphy. *Polynomial selection for the number field sieve integer factorisation algorithm*. Phd thesis, Australian National University, 1999.
38. National Institute of Standards and Technology. Supplemental ITL bulletin for september 2013:NIST opens draft special publication 800-90A, recommendation for random number generation using deterministic random bit generators, for review and comment. http://csrc.nist.gov/publications/nistbul/itlbul2013_09_supplemental.pdf.
39. National Institute of Standards and Technology. Examples for NIST 800-56A, 2006. http://csrc.nist.gov/groups/ST/toolkit/documents/KS_FFC_Prime.pdf.
40. National Institute of Standards and Technology. Digital signature standard (DSS, FIPS-186-4), 2013. Fourth revision.
41. National Institute of Standards and Technology. Recommendation for pair-wise key establishment schemes using discrete logarithm cryptography, SP 800-56A, 2013. Second revision.
42. H. Orman. The Oakley key determination protocol. RFC 2412, Nov. 1998.
43. N. Perlroth, J. Larson, and S. Shane. N.S.A. able to foil basic safeguards of privacy on Web. *The New York Times*, Sep. 5 2013. Online: <http://www.nytimes.com/2013/09/06/us/nsa-foils-much-internet-encryption.html>.
44. C. Pomerance. Analysis and comparison of some integer factoring algorithms. In J. Lenstra, H. W. and R. Tijdeman, editors, *Computational methods in number theory*, volume 154 of *Mathematical Center Tracts*, pages 89–140. Mathematisch Centrum, Amsterdam, 1982.
45. R. Rivest, M. Hellman, J. C. Anderson, and J. W. Lyons. Responses to NIST’s proposal. *CACM*, 35(7):41–54, 1992.
46. O. Schirokauer. Discrete logarithms and local units. *Philos. Trans. Roy. Soc. London Ser. A*, 345(1676):409–423, 1993.
47. M. Scott. Re: NIST announces set of Elliptic Curves. sci.crypt news-group posting dated 1999/06/17, https://groups.google.com/forum/message/raw?msg=sci.crypt/mFMukSsORmI/FpbHDQ6hM_MJ.
48. I. A. Semaev. Special prime numbers and discrete logs in finite prime fields. *Math. Comp.*, 71(237):363–377, 2002.
49. M. E. Smid and D. K. Branstad. Response to comments of the NIST proposed digital signature standard. In E. F. Brickell, editor, *CRYPTO’92*, volume 740 of *Lecture Notes in Comput. Sci.*, pages 76–88. Springer, Heidelberg, Aug. 16–20, 1993.
50. SSL Labs. SSL pulse. <https://www.trustworthyinternet.org/ssl-pulse/>.
51. The CADO-NFS Development Team. CADO-NFS, An Implementation of the Number Field Sieve Algorithm, 2016. Development version (prior to release 2.3), available at <http://cado-nfs.gforge.inria.fr/>.
52. E. Thomé. Subquadratic computation of vector generating polynomials and improvement of the block Wiedemann algorithm. *J. Symbolic Comput.*, 33(5):757–775, Jul. 2002.
53. L. Valenta, D. Adrian, A. Sanso, S. Cohnsey, J. Fried, M. Hastings, J. A. Halderman, and N. Heninger. The most dangerous groups in the world: Exploiting DSA groups for Diffie-Hellman, 2016.

54. A. Young and M. Yung. The dark side of “black-box” cryptography, or: Should we trust capstone? In N. Koblitz, editor, *CRYPTO’96*, volume 1109 of *Lecture Notes in Comput. Sci.*, pages 89–103. Springer, Heidelberg, Aug. 18–22, 1996.

A Complexity analysis of individual logarithms

The complexity analysis of individual logarithms is not well detailed in the literature, in particular in the SNFS case. For convenience we summarize the results in this appendix. As usual in the NFS context, the claimed complexities are not rigorously proven and rely on heuristics.

The notation is the same as in the main body of the paper: p is a prime and we have to compute the discrete logarithm of an element z in a prime order subgroup of \mathbb{F}_p^* . We are given f and g a pair of polynomials that have been used for an NFS computation so that the (virtual) logarithms of all the ideals of norm less than a bound B should have been pre-computed. The bound B , the degrees of f and g , and the sizes of their coefficients depend on the General vs Special NFS variant.

We recall the classical corollary of the Canfield-Erdős-Pomerance theorem that expresses smoothness probabilities in terms of the L -notation:

Theorem 1. *Let a, b, u, v be real numbers such that $a > b > 0$ and $u, v > 0$. As $x \rightarrow \infty$, the proportion of integers below $L_x(a, u)$ that are $L_x(b, v)$ -smooth is*

$$L_x\left(a - b, \frac{u}{v}(a - b) + o(1)\right)^{-1}.$$

A.1 Initialization of the descent

This step consists of first “smoothing” z in order to bootstrap the subsequent descent step. We choose a random integer e , compute the element $z' \equiv z^e \pmod{p}$, and test it for smoothness. Many elements are tested until one is found to be B_{init} -smooth. The best known algorithm for smoothness testing is the ECM algorithm: it extracts (with high probability) all primes up to a bound K in time $L_K(1/2, \sqrt{2} + o(1))$. The dependence in the size of the integer from which we extract primes is polynomial, so we omit it: in our context this type of factor ends up being hidden in the $o(1)$ in the exponents.

From this estimate, one can derive that if we want to allow a running time in $L_p(1/3, \cdot)$, then B_{init} can only be as large as $L_p(2/3, \cdot)$; otherwise, testing the smoothness would be too costly. At the same time, the probability that z' is B_{init} -smooth drives the number of attempts and puts additional constraints. It is remarkable that it also imposes a smoothness bound B_{init} in $L_p(2/3, \cdot)$ to get an $L_p(1/3, \cdot)$ number of attempts. Following Commeine-Semaev [11], if we set $B_{\text{init}} = L_p(2/3, c)$, one can show that the expected running time for the basic algorithm for the initialization step is in $L_p(1/3, \frac{1}{3c} + 2\sqrt{c/3} + o(1))$, which is minimal for $c = 1/\sqrt[3]{3}$, yielding a complexity of $L_p(1/3, \sqrt[3]{3} + o(1)) \approx L_p(1/3, 1.442)$.

Inspired by the early abort strategy that Pomerance [44] had developed in the context of the quadratic sieve, Barbulescu [5] has shown that this complexity can be reduced. The idea is to start the smoothness test with a bound smaller than the target B_{init} smoothness bound: this allows one to extract the smallest factors. Then, we make a decision based on the size of the remaining unfactored part: if it is too large, the probability that this will yield a B_{init} -smooth number is too small and we start again with another random exponent e . In other words, instead of testing immediately for B_{init} -smoothness, we first run a filter, with cheaper ECM parameters, that allows us to select promising candidates for which the full test is run. Analyzing this technique and optimizing all the involved parameters is not a simple task; according to [5], we obtain a final complexity of $\approx L_p(1/3, 1.296)$ for a smoothness bound $B_{\text{init}} = L_p(1/3, 0.771)$.

This is not the end of the story: instead of just one filter, one can add more. The analysis becomes even more involved, but this improves again on the complexity. Numerical experiments indicate that performance does not improve beyond 6 filters, and for 6 filters, the final complexity given in [5] is summarized in the following fact:

Fact. The initialization step of the descent can be done in time $L_p(1/3, 1.232)$ with a smoothness bound $B_{\text{init}} = L_p(1/3, 0.811)$.

Finally, we mention that writing $z' \equiv \frac{u}{v} \pmod{p}$ for u and v that are about half the size of p , and testing them for smoothness does not change the asymptotic complexities, but it yields a huge practical improvement, especially when combined with sieving as in Joux-Lercier [23].

On the other hand, when neither f nor g are linear polynomials, the smoothing test has to be done in one of the number fields, and then, in this context, using half-size elements is necessary to get the appropriate complexity; we refer to [5, Section 8.4.3] for details about this.

A.2 Descent step

After the initialization step, the discrete logarithm of z can be expressed in terms of the virtual logarithms of a few ideals of degree 1 in one of the number fields associated to f or g . Those whose norm is less than the smoothness bound B that was used in the sieving and linear algebra steps are assumed to be already known. Since $B = L_p(1/3, \cdot)$ while $B_{\text{init}} = L_p(2/3, \cdot)$, we expect to have a handful of prime ideals whose logarithms are not known. These are the ones that will be subject to this descent step. We do the analysis for one ideal of maximal size B_{init} ; since there are only polynomially many of them, doing all of them will contribute only to the $o(1)$ in the final exponent.

Let \mathfrak{q} be an ideal of norm $q = L_p(\alpha, c)$, where $\alpha \in [\frac{1}{3}, \frac{2}{3}]$. We consider the lattice of polynomials $\varphi(x) = a_0 + a_1x + \dots + a_{k-1}x^{k-1}$ that, after being mapped to a principal ideal in the number field where \mathfrak{q} belongs, become divisible by \mathfrak{q} . For $k = 2$, this would correspond to the (a, b) -pairs corresponding to \mathfrak{q} seen as a special- q , but we allow larger degrees. Since we are going to allow a search

that takes a time T in $L_p(1/3, \cdot)$ for handling \mathfrak{q} , the a_i 's can be bounded by $(qT)^{1/k} = L_p(\alpha, c/k)L_p(1/3, \cdot)$.

Let us analyze first the case where $\alpha > 1/3$ so that the second factor can be neglected. The product of the norms is given by

$$\begin{aligned} \text{Res}(f(x), \varphi(x)) \text{Res}(g(x), \varphi(x)) &\approx \|\varphi\|^{\deg f + \deg g} \|f\|^{k-1} \|g\|^{k-1} \\ &\approx L_p(\alpha, c/k(\deg f + \deg g)) (\|f\| \|g\|)^{k-1}. \end{aligned}$$

Let us write $\deg f + \deg g = \delta(\log p / \log \log p)^{1/3}$, so that we can cover all the variants. Then $\|f\| \|g\|$ is $L_p(2/3, 2/\delta)$ in the case of GNFS and $L_p(2/3, 1/\delta)$ in the case of SNFS. Finally, the product of the norms is

$$L_p\left(\alpha + 1/3, \frac{c\delta}{k}\right) L_p\left(2/3, \left\{\begin{array}{l} 1 \text{ for SNFS} \\ 2 \text{ for GNFS} \end{array}\right\} \frac{k-1}{\delta}\right).$$

Here there are two strategies: we can fix $k = 2$, so that the second factor does not contribute to the complexity, or we can let k grow in order to balance the two factors.

Descending with higher degree polynomials. The best value for k is proportional to $(\log p / \log \log p)^{\alpha/2 - 1/6}$ (we deliberately omit to analyze the proportionality ratio). In that case, the product of the norms takes the form

$$L_p(\alpha/2 + 1/2, \cdot),$$

so that, since we allow a time $L_p(1/3, \cdot)$, we can expect to find an element that is $L_p(\alpha/2 + 1/6, \cdot)$ -smooth. The smoothness test implies multiplying the cost by $L_p(\alpha/4 + 1/12, \cdot)$, which is bounded by $L_p(1/4, \cdot)$ since $\alpha \leq 2/3$, and therefore does not contribute to the final complexity. As a consequence, as long as α is more than $\frac{1}{3}$, it is possible to descend a \mathfrak{q} whose norm is in $L_p(\alpha, \cdot)$ in prime ideals of norms at most $L_p(\alpha/2 + 1/2, \cdot)$, in time bounded by $L_p(1/3, \cdot)$. We can choose the exponent constant smaller than the other steps of the descent so that these first steps become negligible. This is true whether we are dealing with GNFS or SNFS.

As we get close to $\alpha = \frac{1}{3}$, the value of k tends to a constant. We postpone the corresponding analysis.

Descending with degree-1 polynomials. In the case where we force $k = 2$, the product of the norms is dominated by the first factor and we get $L_p(\alpha + \frac{1}{3}, c\delta/2)$. Let us try to descend \mathfrak{q} in prime ideals of norms slightly smaller than the norm q of \mathfrak{q} , namely we target $L_p(\alpha, c\lambda)$, for some value λ that we hope to be strictly less than 1. The probability of the product of the norms being q^λ -smooth is then in $L_p(\frac{1}{3}, \frac{\delta}{6\lambda} + o(1))^{-1}$. The cost of smoothness testing with ECM is in $L_p(\frac{\alpha}{2}, \cdot)$, which is negligible as soon as $\alpha < 2/3$. Hence, the cost of the descent with degree-1 polynomials is dominated by the case $\alpha = 2/3$, which we will now focus on. In this limiting case, the cost of ECM is $L_{L_p(2/3, c\lambda)}(1/2, \sqrt{2} +$

$o(1) = L_p(1/3, 2\sqrt{c\lambda/3} + o(1))$, so that the time to descend \mathfrak{q} in prime ideals of norms bounded by q^λ is in $L_p(1/3, \frac{\delta}{6\lambda} + 2\sqrt{c\lambda/3} + o(1))$. This is minimized for $\lambda = \sqrt[3]{\delta^2/12c}$ and yields a running time of $L_p(1/3, (3c\delta/2)^{1/3} + o(1))$. In the case of GNFS, we have $\delta = 3^{1/3}$, while it is $\delta = (3/2)^{1/3}$ for SNFS. We fix λ so that we minimize the time when dealing with the largest \mathfrak{q} coming out from the initialization step, namely for $q = L_p(2/3, 0.811)$; this value $c = 0.811$ gives $\lambda = 0.598$ in the case of GNFS, and $\lambda = 0.513$ in the case of SNFS. Both are less than 1, which means that the descent process indeed descends. Finally, we obtain the following:

Fact. If we use degree-1 polynomials, the cost of the first stages of the descent is $L_p(1/3, 1.206)$ for GNFS and $L_p(1/3, 1.117)$ for SNFS.

Last steps of the descent. We now deal with the final steps of the descent where $\mathfrak{q} = L_p(1/3, c)$, with c larger than the constant involved in the smoothness bound $B = L_p(1/3, \cdot)$, which depends on whether we are in the GNFS or SNFS case. In this setting, there is no gain in considering $k > 2$, so we keep $k = 2$. The factor that was neglected when evaluating the size of the a_i 's is no longer negligible, so we start again, and assume that we are going to spend time $T = L_p(1/3, \tau + o(1))$. This propagates into the formulae and gives a bound $L_p(1/3, (\tau + c)/2)$ for the a_i 's, which in turn gives

$$L_p(2/3, (\tau + c)\delta/2) \|f\| \|g\|$$

for the product of the norms. Let us denote $B = L_p(1/3, \beta)$ the smoothness bound used for sieving and linear algebra, and write $c = \beta + \varepsilon$, where $\varepsilon > 0$. We omit the details, but it can be checked that if we allow time $L_p(1/3, \beta)$, we can descend \mathfrak{q} in prime ideals of norms at most $L_p(1/3, \beta + \frac{\varepsilon}{4})$. This analysis is valid both for GNFS and SNFS, even though the values of β and δ are different for these two cases. This is no surprise that this is the cost of finding one relation in the sieving step, since when \mathfrak{q} is just above the smoothness bound, descending involves essentially the same procedure as what we do during sieving with special- q that are marginally smaller. We obtain therefore:

Fact. The cost of the last stages of the descent is $L_p(1/3, 0.961)$ for GNFS and $L_p(1/3, 0.763)$ for SNFS.

In this analysis, we have not studied the transition between the two modes where we decrease the value α or the value c when descending an ideal of size $L_p(\alpha, c)$. This technicality is dealt with in [17] in the context of the Function Field Sieve, but it applies *mutatis mutandis* to our NFS situation.

In the following table, we summarize the exponent constants in the $L_p(1/3, \cdot)$ complexities of the various steps of the descent, for GNFS and SNFS, allowing or not sieving with higher degree polynomials:

Initialization step		Descent step		
		Large q		Small q
		Sieving deg = 1	Sieving higher deg	
GNFS	1.232	1.206	$o(1)$	0.961
SNFS	1.232	1.117	$o(1)$	0.763

B Block Wiedemann algorithm timings

Location	Nodes	CPU Type	Clock Speed	Cores	RAM	Interconnect
UPenn	20	2×Xeon E5-2699v4	2.2 – 2.8GHz	44	512GB	eth40g
	8	2×Xeon E5-2680v3	2.5 – 2.9GHz	24	512GB	eth40g
	6	2×Xeon E5-2699v3	2.3 – 2.8GHz	36	128GB	eth10g
Nancy	48	2×Xeon E5-2650v1	2.0 – 2.4GHz	16	64GB	ib56g

Table 3. We ran both sieving and linear algebra on various clusters of different configurations. For the CPU clock speed, we give both nominal and turbo speeds.

CPU Type	Interconnect	Nodes/Job	Seconds per iteration		
			Sequence	Solution	Communication
Xeon E5-2699v4	eth40g	1		2.42	0.12
		4	0.41		0.17
		8	0.19		0.17
		12	0.13		0.14
		16	0.10	0.21	0.13
Xeon E5-2680v3	eth40g	2		2.24	0.30
		8	0.35		0.15
Xeon E5-2699v3	eth10g	6	0.36		0.33
Xeon E5-2650v1	ib56g	2		3.7	0.19
		8	0.60		0.10

Table 4. Timings for the Block Wiedemann algorithm as run on the various clusters for the 1024-bit SNFS Discrete Log computation. Table 3 gives details on the node configurations.