Co-Z Addition Formulæ and Binary Ladders on Elliptic Curves

Raveen Goundar ▪ Marc Joye ▪ Atsuko Miyaji

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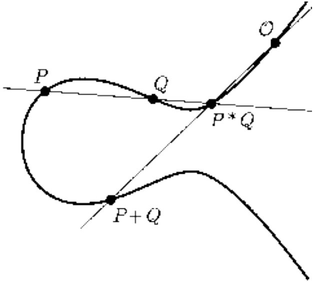
Elliptic Curve Cryptography

- Invented [independently] by Neil Koblitz and Victor Miller in 1985



- Useful for key exchange, encryption and digital signature

Scalar Multiplication

Definition

Given scalar k and a point $\boldsymbol{P}$, compute $[k]\boldsymbol{P} = \underbrace{\boldsymbol{P} + \boldsymbol{P} + \cdots + \boldsymbol{P}}_{k \text{ times}}$

ECDLP Given $\boldsymbol{P}$ and $\boldsymbol{Q} = [k]\boldsymbol{P}$, recover k

- no subexponential algorithms are known to solve the ECDLP (in the *general* case)
- smaller key sizes can be used

	Bit security				
	80	112	128	192	256
ECC	160	224	256	384	512
RSA	1024	2048	3072	8192	15360

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This Talk

Goal

Implementation of the Montgomery ladder and of its dual version using efficient co-Z formulæ

- binary scalar multiplication algorithms
- suitable for memory-constrained devices

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Outline

1. **Arithmetic on Elliptic Curves**
   - Jacobian coordinates
   - Co-Z point addition
2. **Binary Scalar Multiplication Algorithms**
   - Left-to-right methods
   - Right-to-left methods
3. **New Implementations**
   - Binary ladders with co-Z trick
   - Point double-add operation
4. **Discussion**
   - Performance analysis
   - Security analysis
5. **Conclusion**

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1 **Arithmetic on Elliptic Curves**
- Jacobian coordinates
- Co-Z point addition

2 **Binary Scalar Multiplication Algorithms**
- Left-to-right methods
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3 **New Implementations**
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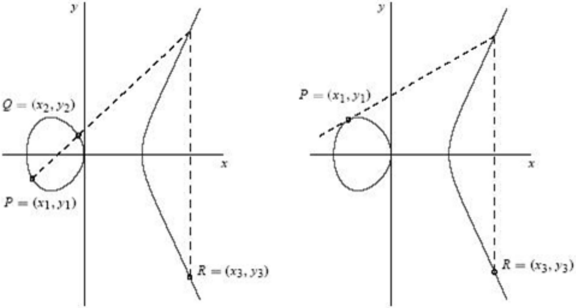
5 **Conclusion**

Elliptic Curves

Weierstraß equation (affine coordinates)

Let $E : y^2 = x^3 + ax + b$ define over $\mathbb{F}_q$ ($char \neq 2, 3$) with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$



(a) Addition: $P + Q = R$. (b) Doubling: $P + P = R$.

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Jacobian Coordinates

- To avoid computation of inverse in $\mathbb{F}_q$
  - affine point $(x, y) \rightarrow$ projective point $(X : Y : Z)$ such that $x = X/Z^2$ and $y = Y/Z^3$

Weierstraß equation (projective Jacobian coordinates)

Let $E : Y^2 = X^3 + aXZ^4 + bZ^6$ define over $\mathbb{F}_q$ $(char \neq 2, 3)$ with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$

- Point at infinity $\boldsymbol{O} = (1 : 1 : 0)$
- If $\boldsymbol{P} = (X_1 : Y_1 : Z_1) \in E$ then $-\boldsymbol{P} = (X_1 : -Y_1 : Z_1)$

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Co-Z Point Addition (ZADD)

- Introduced by Meloni [WAIFI 2007]
- Addition of two distinct points with the same Z-coordinate

Co-Z point addition

Let $\boldsymbol{P} = (X_1 : Y_1 : Z)$ and $\boldsymbol{Q} = (X_2 : Y_2 : Z)$. Then $\boldsymbol{P} + \boldsymbol{Q} = (X_3 : Y_3 : Z_3)$ where

$$X_3 = D - W_1 - W_2, \quad Y_3 = (Y_1 - Y_2)(W_1 - X_3) - A_1, \quad Z_3 = Z(X_1 - X_2)$$

with $A_1 = Y_1(W_1 - W_2)$, $W_1 = X_1 C$, $W_2 = X_2 C$, $C = (X_1 - X_2)^2$ and $D = (Y_1 - Y_2)^2$

- Cost of ZADD: $5\mathrm{M} + 2\mathrm{S}$

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Co-Z Point Addition with Update (ZADDU)

- Main advantage of Meloni's addition

Equivalent representation of P

Evaluation of $\boldsymbol{R} = \mathrm{ZADD}(\boldsymbol{P}, \boldsymbol{Q})$ yields for free

$$\boldsymbol{P}' = \left(X_1(X_1 - X_2)^2 : Y_1(X_1 - X_2)^3 : Z_3\right) = (W_1 : A_1 : Z_3) \sim \boldsymbol{P}$$

that is, $\mathrm{Z}(\boldsymbol{P}') = \mathrm{Z}(\boldsymbol{R})$

- Notation: $(\boldsymbol{R}, \boldsymbol{P}') = \mathrm{ZADDU}(\boldsymbol{P}, \boldsymbol{Q})$
- Cost of ZADDU: $\underline{5\mathrm{M} + 2\mathrm{S}}$

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Left-to-Right Methods

Algorithm 1 Left-to-right binary method

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{O}$; $\boldsymbol{R_1} \leftarrow \boldsymbol{P}$
2: **for** $i = n-1$ down to 0 **do**
3: $\quad \boldsymbol{R_0} \leftarrow 2\boldsymbol{R_0}$
4: $\quad$ **if** $(k_i = 1)$ **then** $\boldsymbol{R_0} \leftarrow \boldsymbol{R_0} + \boldsymbol{R_1}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

Algorithm 2 Montgomery ladder

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{O}$; $\boldsymbol{R_1} \leftarrow \boldsymbol{P}$
2: **for** $i = n-1$ down to 0 **do**
3: $\quad b \leftarrow k_i$; $\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
4: $\quad \boldsymbol{R_b} \leftarrow 2\boldsymbol{R_b}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

- Subject to SPA-type attacks
- Inserting dummy addition prevents SPA
  - subject to safe-error attacks

- Regular structure, no dummy operations
- Naturally resistant against SPA and safe-error attacks
- 2 registers

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Right-to-Left Methods

Algorithm 3 Right-to-left binary method

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{O}$; $\boldsymbol{R_1} \leftarrow \boldsymbol{P}$
2: **for** $i = 0$ to $n - 1$ **do**
3:     **if** $(k_i = 1)$ **then** $\boldsymbol{R_0} \leftarrow \boldsymbol{R_0} + \boldsymbol{R_1}$
4:     $\boldsymbol{R_1} \leftarrow 2\boldsymbol{R_1}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

- Idem left-to-right method

  (SPA-type attacks, safe-error attacks)

Algorithm 4 Joye's double-add

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{O}$; $\boldsymbol{R_1} \leftarrow \boldsymbol{P}$
2: **for** $i = 0$ to $n - 1$ **do**
3:     $b \leftarrow k_i$
4:     $\boldsymbol{R_{1-b}} \leftarrow 2\boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

- Idem Montgomery ladder

  (regular structure, no dummy operations, 2 registers)

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Conjugate co-Z Point Addition (ZADDC)

- New co-Z point operation
  - using caching techniques

Conjugate co-Z point addition

From $-\boldsymbol{Q} = (X_2 : -Y_2 : Z_2)$, evaluation of $\boldsymbol{R} = \mathrm{ZADD}(\boldsymbol{P}, \boldsymbol{Q})$ allows one to get $\boldsymbol{S} := \boldsymbol{P} - \boldsymbol{Q} = (\overline{X_3}, \overline{Y_3}, Z_3)$ where

$$\overline{X_3} = (Y_1 + Y_2)^2 - W_1 - W_2, \quad \overline{Y_3} = (Y_1 + Y_2)(W_1 - \overline{X_3})$$

with an additional cost of $1\mathrm{M} + 1\mathrm{S}$

- Notation: $(\boldsymbol{P} + \boldsymbol{Q}, \boldsymbol{P} - \boldsymbol{Q}) = \mathrm{ZADDC}(\boldsymbol{P}, \boldsymbol{Q})$
- Total cost of ZADDC: $\underline{6\mathrm{M} + 3\mathrm{S}}$

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Left-to-Right Binary Ladder With co-Z Trick

Algorithm 5 Montgomery ladder with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_{n-1} = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

$\boldsymbol{R_0} \leftarrow \boldsymbol{O}$; $\boldsymbol{R_1} \leftarrow \boldsymbol{P}$
for $i = n - 1$ down to 0 **do**
    $b \leftarrow k_i$; $\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
    $\boldsymbol{R_b} \leftarrow 2\boldsymbol{R_b}$
end for
return $\boldsymbol{R_0}$

Left-to-Right Binary Ladder With co-Z Trick

Algorithm 5 Montgomery ladder with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_{n-1} = 1$

Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_1}, \boldsymbol{R_0}) \leftarrow \mathrm{DBLU}(\boldsymbol{R_0})$
2: **for** $i = n-2$ down to 0 **do**
3: $\quad b \leftarrow k_i$; $\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
4: $\quad \boldsymbol{R_b} \leftarrow 2\boldsymbol{R_b}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

$(2\boldsymbol{P}, \boldsymbol{P'}) = \mathrm{DBLU}(\boldsymbol{P})$ where $\boldsymbol{P'} \sim \boldsymbol{P}$ and $\mathrm{Z}(\boldsymbol{P'}) = \mathrm{Z}(2\boldsymbol{P})$
Cost: $1\mathrm{M} + 5\mathrm{S}$

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Left-to-Right Binary Ladder With co-Z Trick

Algorithm 5 Montgomery ladder with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_{n-1} = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_1}, \boldsymbol{R_0}) \leftarrow \mathrm{DBLU}(\boldsymbol{R_0})$
2: **for** $i = n-2$ down to 0 **do**
3: $\quad b \leftarrow k_i$; $\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
4: $\quad \boldsymbol{R_b} \leftarrow 2\boldsymbol{R_b}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

$\boldsymbol{T} \leftarrow \boldsymbol{R_b} - \boldsymbol{R_{1-b}}$
$\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{R_b} + \boldsymbol{R_{1-b}}$; $\boldsymbol{R_b} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{T}$ $(= 2\boldsymbol{R_b})$

Left-to-Right Binary Ladder With co-Z Trick

Algorithm 5 Montgomery ladder with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_{n-1} = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_1}, \boldsymbol{R_0}) \leftarrow \mathrm{DBLU}(\boldsymbol{R_0})$
2: **for** $i = n-2$ down to 0 **do**
3: $\quad b \leftarrow k_i$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{ZADDC}(\boldsymbol{R_b}, \boldsymbol{R_{1-b}})$
4: $\quad (\boldsymbol{R_b}, \boldsymbol{R_{1-b}}) \leftarrow \mathrm{ZADDU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$
5: **end for**
6: **return** $\boldsymbol{R_0}$

$\boldsymbol{T} \leftarrow \boldsymbol{R_b} - \boldsymbol{R_{1-b}}$
$\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{R_b} + \boldsymbol{R_{1-b}}$; $\boldsymbol{R_b} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{T}$ $(= 2\boldsymbol{R_b})$

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Left-to-Right Binary Ladder With co-Z Trick

Algorithm 5 Montgomery ladder with co-Z formulæ

Input: $P \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_{n-1} = 1$
Output: $Q = kP$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_1}, \boldsymbol{R_0}) \leftarrow \mathrm{DBLU}(\boldsymbol{R_0})$
2: **for** $i = n-2$ down to 0 **do**
3: $\quad b \leftarrow k_i$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{ZADDC}(\boldsymbol{R_b}, \boldsymbol{R_{1-b}})$
4: $\quad (\boldsymbol{R_b}, \boldsymbol{R_{1-b}}) \leftarrow \mathrm{ZADDU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$
5: **end for**
6: **return** $\boldsymbol{R_0}$

- Cost per bit: $(6\mathrm{M} + 3\mathrm{S}) + (5\mathrm{M} + 2\mathrm{S}) = 11\mathrm{M} + 5\mathrm{S}$

Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{O}$; $\boldsymbol{R_1} \leftarrow \boldsymbol{P}$
2: **for** $i = 0$ to $n-1$ **do**
3: $\quad b \leftarrow k_i$;
4: $\quad \boldsymbol{R_{1-b}} \leftarrow 2\boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

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Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $\boldsymbol{R_0} \leftarrow \boldsymbol{P}$; $\boldsymbol{R_1} \leftarrow \boldsymbol{P}$
2: **for** $i = 1$ to $n-1$ **do**
3: $\quad b \leftarrow k_i$;
4: $\quad \boldsymbol{R_{1-b}} \leftarrow 2\boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

$\boldsymbol{R_0}$ and $\boldsymbol{R_1}$ now have the same Z-coordinate but are not different (!)
$\Longrightarrow$ start for-loop at $i = 2$

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Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

```
b ← k_1; R_b ← P; (R_{1−b}, R_b) ← TPLU(R_b)
for i = 2 to n − 1 do
    b ← k_i;
    R_{1−b} ← 2R_{1−b} + R_b
end for
return R_0
```

$(3\boldsymbol{P}, \boldsymbol{P}') = \mathrm{TPLU}(\boldsymbol{P})$ where $\boldsymbol{P}' \sim \boldsymbol{P}$ and $\mathrm{Z}(\boldsymbol{P}') = \mathrm{Z}(3\boldsymbol{P})$
Cost: $6\mathrm{M} + 7\mathrm{S}$

Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $b \leftarrow k_1$; $\boldsymbol{R_b} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{TPLU}(\boldsymbol{R_b})$
2: **for** $i = 2$ to $n-1$ **do**
3: $\quad b \leftarrow k_i$;
4: $\quad \boldsymbol{R_{1-b}} \leftarrow 2\boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
5: **end for**
6: **return** $\boldsymbol{R_0}$

Can be rewritten as $\boldsymbol{T} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{R_b}$; $\quad \boldsymbol{R_{1-b}} \leftarrow \boldsymbol{T} + \boldsymbol{R_{1-b}}$

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Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$

Output: $\boldsymbol{Q} = k\boldsymbol{P}$

$b \leftarrow k_1$; $\boldsymbol{R_b} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{TPLU}(\boldsymbol{R_b})$
for $i = 2$ to $n-1$ **do**
  $b \leftarrow k_i$; $\boldsymbol{T} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
  $\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{T} + \boldsymbol{R_{1-b}}$
end for
return $\boldsymbol{R_0}$

$(\boldsymbol{T}, \boldsymbol{R_{1-b}}) \leftarrow \mathrm{ZADDU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$; $(\boldsymbol{R_{1-b}}, \boldsymbol{T}) \leftarrow \mathrm{ZADDU}(\boldsymbol{T}, \boldsymbol{R_{1-b}})$

+ update of $\boldsymbol{R_b}$ (cost: 3M)

Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

$b \leftarrow k_1$; $\boldsymbol{R_b} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{TPLU}(\boldsymbol{R_b})$
for $i = 2$ to $n-1$ **do**
    $b \leftarrow k_i$; $\boldsymbol{T} \leftarrow \boldsymbol{R_{1-b}} + \boldsymbol{R_b}$
    $\boldsymbol{R_{1-b}} \leftarrow \boldsymbol{T} + \boldsymbol{R_{1-b}}$
end for
return $\boldsymbol{R_0}$

$(\boldsymbol{T}, \boldsymbol{R_{1-b}}) \leftarrow \mathrm{ZADDU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{ZADDC}(\boldsymbol{T}, \boldsymbol{R_{1-b}})$
since $\boldsymbol{T} - \boldsymbol{R_{1-b}} = \boldsymbol{R_b}$

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Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $b \leftarrow k_1$; $\boldsymbol{R_b} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{TPLU}(\boldsymbol{R_b})$
2: **for** $i = 2$ to $n-1$ **do**
3: $\quad b \leftarrow k_i$; $(\boldsymbol{R_b}, \boldsymbol{R_{1-b}}) \leftarrow \mathrm{ZADDU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$
4: $\quad (\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{ZADDC}(\boldsymbol{R_b}, \boldsymbol{R_{1-b}})$
5: **end for**
6: **return** $\boldsymbol{R_0}$

$(\boldsymbol{T}, \boldsymbol{R_{1-b}}) \leftarrow \mathrm{ZADDU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{ZADDC}(\boldsymbol{T}, \boldsymbol{R_{1-b}})$
since $\boldsymbol{T} - \boldsymbol{R_{1-b}} = \boldsymbol{R_b}$

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Right-to-Left Binary Ladder With co-Z Trick

Algorithm 6 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

1: $b \leftarrow k_1$; $\boldsymbol{R_b} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{TPLU}(\boldsymbol{R_b})$
2: **for** $i = 2$ to $n-1$ **do**
3: $\quad b \leftarrow k_i$; $(\boldsymbol{R_b}, \boldsymbol{R_{1-b}}) \leftarrow \mathrm{ZADDU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$
4: $\quad (\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{ZADDC}(\boldsymbol{R_b}, \boldsymbol{R_{1-b}})$
5: **end for**
6: **return** $\boldsymbol{R_0}$

- Cost per bit: $(5\mathrm{M} + 2\mathrm{S}) + (6\mathrm{M} + 3\mathrm{S}) = \underline{11\mathrm{M} + 5\mathrm{S}}$

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Point Doubling-Addition

- Point doubling-addition evaluates: $\boldsymbol{R} \leftarrow 2\boldsymbol{P} + \boldsymbol{Q}$
  - $\boldsymbol{T} \leftarrow \boldsymbol{P} + \boldsymbol{Q}$ followed by $\begin{cases} \boldsymbol{R} \leftarrow \boldsymbol{T} + \boldsymbol{P} \\ \boldsymbol{Q} \leftarrow \boldsymbol{T} - \boldsymbol{P} \end{cases}$
  - $(\boldsymbol{T}, \boldsymbol{P}) \leftarrow \mathrm{ZADDU}(\boldsymbol{P}, \boldsymbol{Q})$; $(\boldsymbol{R}, \boldsymbol{Q}) \leftarrow \mathrm{ZADDC}(\boldsymbol{T}, \boldsymbol{P})$
  - cost: $11\mathrm{M} + 5\mathrm{S}$
- Combined operation

Co-Z point doubling-addition with update

$$(\boldsymbol{R}, \boldsymbol{Q}) \leftarrow \mathrm{ZDAU}(\boldsymbol{P}, \boldsymbol{Q})$$

  - trades 2M against 2S
  - cost: $\underline{9\mathrm{M} + 7\mathrm{S}}$

Application

Algorithm 7 Joye's double-add with co-Z formulæ

Input: $\boldsymbol{P} \in E(\mathbb{F}_q)$ and $k = (k_{n-1}, \ldots, k_0)_2 \in \mathbb{N}$ with $k_0 = 1$
Output: $\boldsymbol{Q} = k\boldsymbol{P}$

$b \leftarrow k_1$; $\boldsymbol{R_b} \leftarrow \boldsymbol{P}$; $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{TPLU}(\boldsymbol{R_b})$
for $i = 2$ to $n-1$ **do**
    $b \leftarrow k_i$
    $(\boldsymbol{R_{1-b}}, \boldsymbol{R_b}) \leftarrow \mathrm{ZDAU}(\boldsymbol{R_{1-b}}, \boldsymbol{R_b})$
end for
return $\boldsymbol{R_0}$

- Cost per bit: $9\mathrm{M} + 7\mathrm{S}$
- (Similar saving applies to Montgomery ladder)

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- Point double-add operation

4 **Discussion**
- **Performance analysis**
- **Security analysis**

5 Conclusion

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Performance: Addition Formulæ

Operation	Notation	Cost
Point addition:		
– general addition	ADD	$11M + 5S$
– co-Z addition	ZADD	$5M + 2S$
– co-Z addition with update	ZADDU	$\underline{5M + 2S}$
– general conjugate addition	ADDC	$12M + 6S$
– conjugate co-Z addition	ZADDC	$\underline{6M + 3S}$
Point doubling-addition:		
– general version	DA	$13M + 8S$
– mixed version	mDA	$11M + 7S$
– co-Z version with update	ZDAU	$\underline{9M + 7S}$

- Comparison
  - very efficient co-Z formulæ

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Performance: Scalar Multiplication

Algorithm	Operations	Cost per bit
Joye's double-add:		
— basic version	DA	13M + 8S
— co-Z version	ZDAU	9M + 7S
Montgomery ladder:		
— basic version	DBL and ADD	14M + 10S
— X-only version	XDBL and XADD	9M + 7S †
— co-Z version	ZDAU′	9M + 7S

† assuming that multiplications by a have negligible cost

- Comparison
  - co-Z versions are always faster
  - cost is independent of the curve parameters
- Latest news: cost reduced to 8M + 6S with new ZACAU′ op.

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Performance: Scalar Multiplication

Algorithm	Operations	Cost per bit
Joye's double-add:		
— basic version	DA	$13M + 8S$
— co-Z version	ZDAU	$9M + 7S$
Montgomery ladder:		
— basic version	DBL and ADD	$14M + 10S$
— X-only version	XDBL and XADD	$9M + 7S$ †
— co-Z version	ZACAU′	$8M + 6S$

† assuming that multiplications by a have negligible cost

- Comparison
  - co-Z versions are always faster
  - cost is independent of the curve parameters
- Latest news: cost reduced to $8M + 6S$ with new ZACAU′ op.

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Security Analysis

- Proposed co-Z implementations are built on highly regular scalar multiplication algorithms
  - inherit similar security features
  - naturally resistant against
    - SPA-type attacks
    - safe-error attacks
- Can be combined with existing DPA-type countermeasures
- Output complete point representation
  - possible to check redundant relations
    - e.g., output point belongs to the curve
  - useful feature against (regular) fault attacks

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Outline

1. **Arithmetic on Elliptic Curves**
   - Jacobian coordinates
   - Co-Z point addition
2. **Binary Scalar Multiplication Algorithms**
   - Left-to-right methods
   - Right-to-left methods
3. **New Implementations**
   - Binary ladders with co-Z trick
   - Point double-add operation
4. **Discussion**
   - Performance analysis
   - Security analysis
5. **Conclusion**

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Summary

- New strategies for evaluating scalar multiplications on elliptic curves using co-Z arithmetic
  - nicely combine with certain binary ladders
- Efficient co-Z conjugate point addition formula (as well as other companion co-Z formulæ)
  - require 7 or 8 registers
  - suitable for memory constrained devices

Full version available at http://eprint.iacr.org/2010/309

Acknowledgments

- We would like to thank
  - Jean-Luc Beuchat
  - Francisco Rodrìguez Henrìquez
  - Patrick Longa
  - Francesco Sica
  - Alexandre Venelli
  - An anonymous referee

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Questions?

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