Range Extension for Weak PRFs

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(weak) pseudorandom functions

$$\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots\}, \mathcal{F}_n : \mathcal{K}_n \times \mathcal{X}_n \to \mathcal{Y}_n$$

is a pseudorandom function (PRF) if

- $F(k, x)$ can be efficiently computed.
- $F(k, .)$ (with a random key $k \in \mathcal{K}_n$) cannot be efficiently distinguished from a uniformly random function $\mathcal{R}$.

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is a weak pseudorandom function (wPRF) if

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wPRFs are weaker primitives than PRFs, so relying on the security of a block-cipher like AES as a wPRF is more secure than assuming it to be a PRF.

black-box range extension

Let C be a circuit with oracle gates, such that for any

$$F : \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$$

we have

$$C_F : \mathcal{K}^t \times \{0,1\}^{n'} \to \{0,1\}^{n \cdot e}$$

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Definition

C is a secure range extension for wPRFs, if for any wPRFs F, also C_F is wPRF.

applications

For a wPRF F and a secure expansion C, (Enc, Dec) as below is a secure encryption scheme.

$Enc(k, M):$ sample X at random and output $(C_F(k, X) \oplus M, X)$

$Dec(k, (C, X)):$ output $C_F(k, X) \oplus C$.

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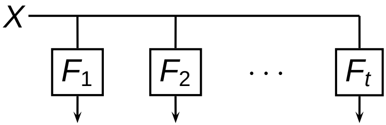
Overhead just one block. Key length depends on the key-expansion of C_F.

example 1: parallel evaluation

$$C_F(\{k_1, \ldots, k_t\}, X) = F(k_1, X), \ldots, F(k_t, X)$$



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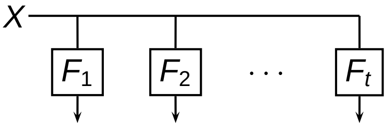

+ Secure range extension for PRF and wPRF.

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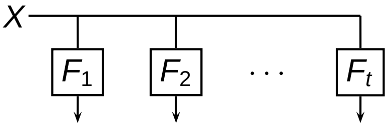
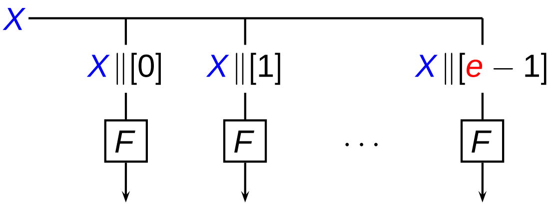


+ Secure range extension for PRF and wPRF.
– Range expansion = Key expansion (very low).

example 2: parallel evaluation with one key

$$C_F(k, X) = F(k, X \| [0]), \ldots, F(k, X \| [e-1])$$

$e = 2^z$, $X \in \{0,1\}^{n-z}$
$[i]$ is binary representation of $[i]$ padded to length z.



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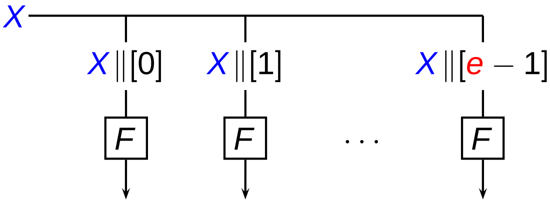

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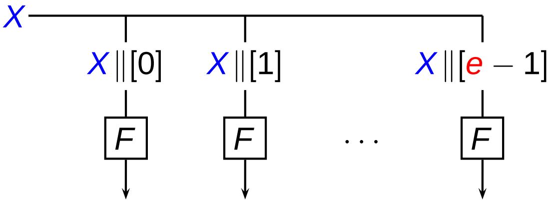

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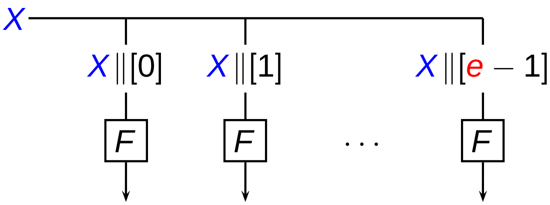
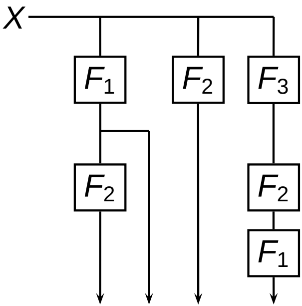


+ Just one key.
+ Secure range extension for PRF.
– Not Secure range extension for wPRF.
  E.g. for a wPRF where $F(k, X\|[0]) = F(k, X\|[1])$.

a general class of range extensions



$C_F^{[1,12,2,321]}$

a general class of range extensions

Definition

Let $s = \{s_1, \ldots, s_e\}$, each $s_i \in \{1, \ldots, t\}^*$. Define

$$C_F^s(k_1, \ldots, k_t, X) = Y_1, \ldots, Y_e$$

where Y_i is computed by applying F on input X sequentially as defined by s_i, i.e. with $m = |s_i|$

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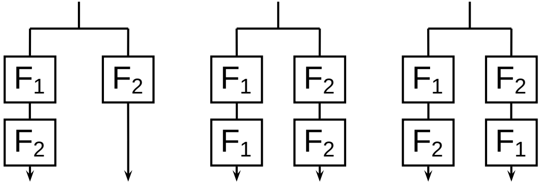
All known (efficient) secure range expansion for wPRFs are of this form (like in the previous talk).
For which s is C^s a secure range expansion for wPRFs?

The Good, the Bad and the Ugly [1]

Which of $C^{[12,2]}$, $C^{[11,22]}$, $C^{[12,21]}$ is a secure range extension for wPRFs?



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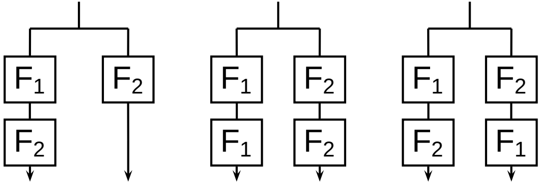

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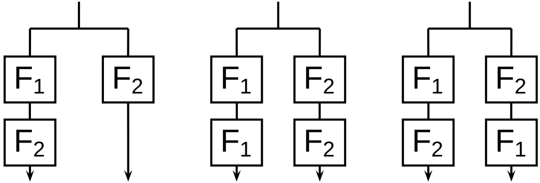

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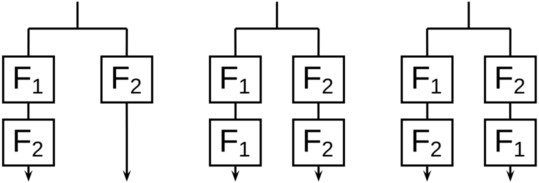

- $C^{[12,2]}$ is secure via a black-box reduction.
- $C^{[11,22]}$ is not secure via a black-box reduction.
- $C^{[12,21]}$ cannot be proven secure nor insecure via a black-box reduction.

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- $C^\alpha, \alpha \subset \mathbb{N}^*$ is good if the security of C^α (as range expansion for wPRFs) can be proven via a black-box reduction.

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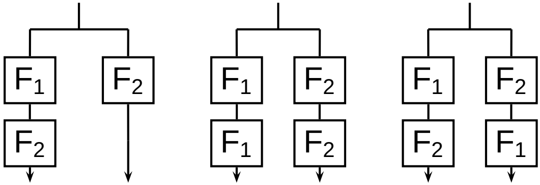
- C^{α} is ugly if it's not good and not bad.

We completely classify C^{α} (as good, bad or ugly) by simple properties of α.

Theorem (Complete Classification)

$C^\alpha, \alpha = \{s_1, \ldots, s_t\}$ *is*

- *bad if α contains a string with two consecutive identical letters or two identical strings.*
- *good if it's not bad and whenever a letter c appears before a letter d in some $s \in \alpha$, then d does not appear before c in any string $s' \in \alpha$.*
- *ugly if it's not good nor bad.*



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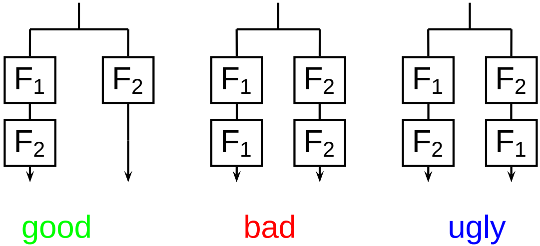
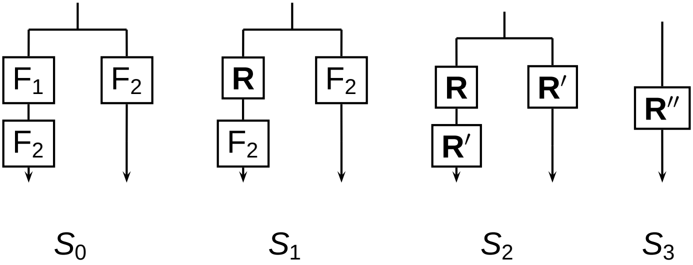
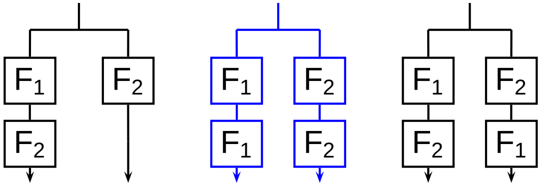
We sketch the proof only for our three special cases:



The Good: Security via Black-Box Reduction



- $S_0 \rightarrow S_1$ safe replacement.
- $S_1 \rightarrow S_2$ safe replacement.
- $\Delta_q^{\mathrm{KPA}}(S_2, S_3) \leq q^2/|Range|$



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The Bad: Black-Box Counterexample

For a pseudorandom permutation* G define H^{G} :

- if $X = 0\dots0$ then $\mathsf{H}^{\mathsf{G}}(k, X) = 0\dots0$
- Otherwise, let $Y = {}_LY \|_R Y = G^{-1}(k, X)$.

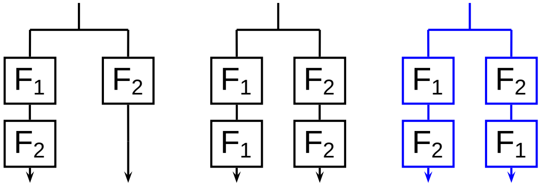
$$\mathsf{H}^{\mathsf{G}}(X) = \begin{cases} 0\dots0 & \text{if } {}_LY = 0\dots0 \\ \mathsf{G}(k, 0\dots0\|_R X) & \text{otherwise} \end{cases}$$

Lemma

$\mathsf{H}^{\mathsf{G}}(k, .)$ *is a wPRF but* $\mathsf{H}^{\mathsf{G}}(k, \mathsf{H}^{\mathsf{G}}(k, .))$ *is not.*

$$X \longrightarrow \boxed{\mathsf{H}^{\mathsf{G}}(k, .)} \xrightarrow[G(k, 0\dots0\|_R X)]{} \boxed{\mathsf{H}^{\mathsf{G}}(k, .)} \longrightarrow 0\dots0$$

*A PRP can be constructed from a wPRF via a black-box reduction (GMM then Luby-Rackoff)



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- A black-box reduction holds relative to any oracle.
- So to show $C^{[12,21]}$ is not good we must come up with an oracle $\mathcal{O}$ such that
  - relative to $\mathcal{O}$ wPRFs $F^{\mathcal{O}}$ exist
  - $C^{[12,21]}_{F^{\mathcal{O}}}$ is not a wPRF.

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- Similarly, to show $C^{[12,21]}$ is not bad we must come up with an oracle $\mathcal{O}$ such that relative to $\mathcal{O}$ $C^{[12,21]}_{F^{\mathcal{O}}}$ is a wPRF for any wPRF $F^{\mathcal{O}}$.

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The Ugly: Insecure under DDH

$G = \langle g \rangle$: prime order cyclic group where DDH is hard, then for random $x \in \mathbb{Z}_{|G|}$

$$a \longrightarrow \boxed{\mathrm{F}(x,.)} \longrightarrow a^x$$

is a wPRF, but $C_F^{[12,21]}$

$$\begin{tikzcd} & \boxed{\mathrm{F}(x,.)} \arrow[r, dash] & a^x \arrow[r, dash] & \boxed{\mathrm{F}(y,.)} \arrow[r] & a^{xy} \\ a \arrow[ur, dash] \arrow[dr, dash] & & & & \\ & \boxed{\mathrm{F}(y,.)} \arrow[r, dash] & a^y \arrow[r, dash] & \boxed{\mathrm{F}(x,.)} \arrow[r] & a^{xy} \end{tikzcd}$$

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Questions?