LEFTOVER HASH LEMMA

REVISITED

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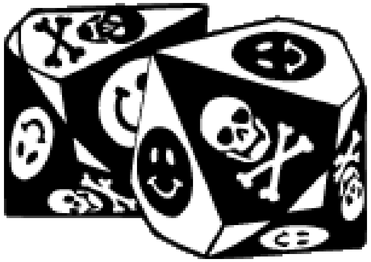
Imperfect Random Sources



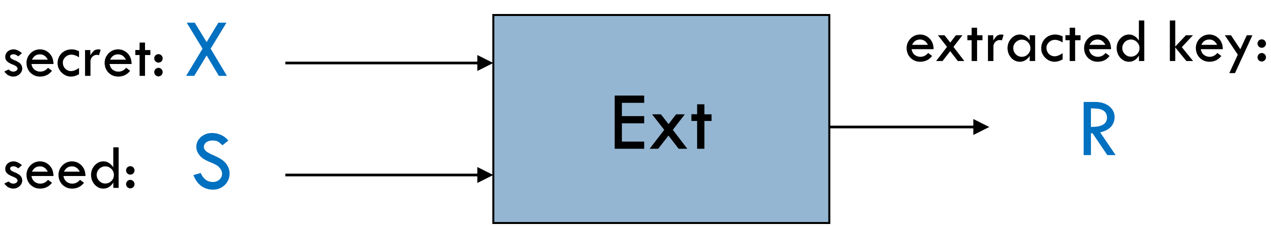
- Ideal randomness is crucial in many areas
  - Especially cryptography (i.e., secret keys) [MP91,DOPS04,BD07]
- However, often deal with imperfect randomness
  - physical sources, biometric data, partial knowledge about secrets, extracting from group elements (DH key exchange),…
- Necessary assumption: must have (min-)entropy
  - (Min-entropy) m-source: $\Pr[X=x] \le 2^{-m}$, for all x
- Can we extract (nearly) perfect randomness from such realistic, imperfect sources?

Extractors

- Tool: Randomness Extractor [NZ96].
  - Input: a *weak secret* X and a *uniformly random seed* S.
  - Output: *extracted key* R = **Ext**(X; S).
  - R is uniformly random, even conditioned on the seed S.

$$(\mathbf{Ext}(X; S), S) \approx (\text{Uniform}, S)$$

- **Many uses in complexity theory and cryptography.**
  - Well beyond key derivation (de-randomization, etc.)



Parameters

- Min-entropy m.
- Output length v.
  - Equivalent measure: Entropy Loss $L = m - v$.
- Error ε (measures *statistical* distance from uniform).
  - Defines security parameter $k = \log(1/\varepsilon)$
- Seed Length n.
- <u>Optimal Parameters</u> [Sip, RT, DO]:
  - Seed length $n = \mathrm{O}(\text{security parameter } \log(1/\varepsilon))$
  - Entropy loss $L = 2\log(1/\varepsilon)$
- Can we match them <u>efficiently</u>?

Leftover Hash Lemma (LHL)

"Todays special is yesterdays left overs."

Leftover Hash Lemma (LHL)

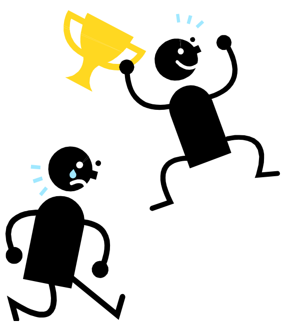
- Universal Hash Family $\mathcal{H} = \{ h\colon \mathfrak{X} \to \{0,1\}^v \}$:

$$\forall\, x \neq y,\ \ \Pr_h[\, h(x) = h(y)\,] = \frac{1}{2^v}$$

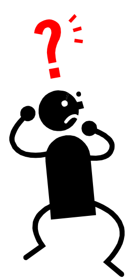
- Leftover Hash Lemma [HILL].

  Universal hash functions $\{\boldsymbol{h}\}$ yield good extractors:

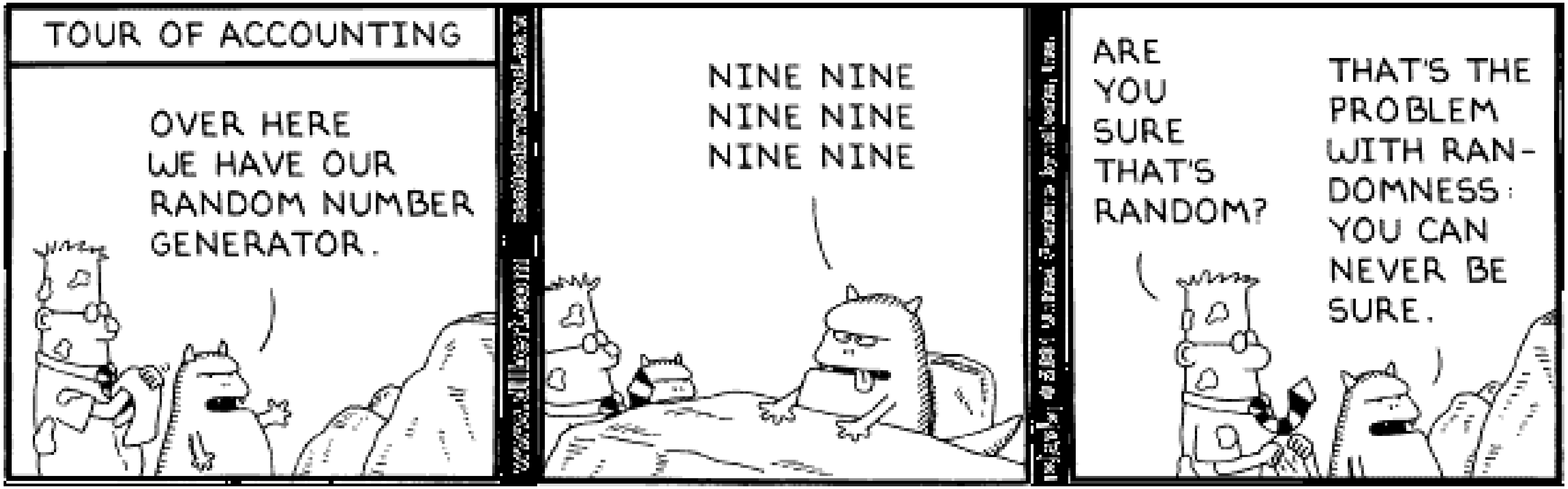
$$(h\,(\mathrm{X}),\, h) \approx_\varepsilon (U_v\,,\, h)$$



  - optimal entropy loss: $L = 2\log(1/\varepsilon)$
  - sub-optimal seed length: $n \geq |\mathrm{X}|$
- Pros: simple, <u>very fast</u>, nice algebraic properties
- Cons: large seed and entropy loss



Part I: Improving the Entropy Loss



Is it Important?

- Yes! Many sources do not have "extra" $2\log(1/\varepsilon)$ bits
  - Biometrics, physical sources, DH keys of elliptic curves (EC)
    - DH: lower "start-up" min-entropy also improves efficiency
- Heuristic extractors, analyzed in the random oracle model, have "no entropy loss"
- <u>End Result</u>: practitioners prefer heuristic key derivation to provable key derivation (see [DGH$^+$,Kra])
- <u>Goal</u>: provably reduce $2\log(1/\varepsilon)$ entropy loss of LHL closer to "no entropy loss" of heuristic extractors

Is not $2\log(1/\varepsilon)$ entropy loss optimal?

- Yes, if must protect against all distinguishers D
- Cryptographic Setting: restricted distinguishers D
  - D = combination of attacker A and challenger C
  - D outputs 1 ⇔ A won the game against C
- Case Study: key derivation for signature/MAC
  - Assume: Pr[A forges sig with random key] $\leq \varepsilon$ (= negl)
  - Hope: Pr[A forges sig with extracted key] $\leq \varepsilon'$ ($\approx \varepsilon$)
  - Key Insight: only care about distinguishers which almost never succeed (on uniform keys) in the first place!
  - Better entropy loss might be possible!

Improved Entropy Loss for Key Derivation

- Setting: application P needs a v–bit secret key R
  - Ideal Model: $R \leftarrow U_v$ is uniform
  - Real Model: $R \leftarrow \mathrm{Ext}(X; S)$, where $H_\infty(X) = v + L$
- Assumption: P is ε–secure in the ideal model
- Conclusion: P is ε'–secure in the real model
- **Standard LHL**: if Ext is universal hash function, then

$$\varepsilon' \leq \varepsilon + \sqrt{2^{-L}}$$

- **Our Result**: For a "wide range" of applications P

$$\varepsilon' \leq \varepsilon + \sqrt{\varepsilon \cdot 2^{-L}}$$

Improved Entropy Loss for Key Derivation

Moral:

Might extract more if know why you are extracting

- **Standard LHL**: if Ext is universal hash function, then

$$\varepsilon' \le \varepsilon + \sqrt{2^{-L}}$$

- **Our Result**: For a "wide range" of applications P

$$\varepsilon' \le \varepsilon + \sqrt{\varepsilon \cdot 2^{-L}}$$

Comparison

- **<u>Standard LHL</u>**: $\varepsilon' \leq \varepsilon + \sqrt{2^{-L}}$
  - Must have $L \geq 2\log(1/\varepsilon)$ for $\varepsilon' = 2\varepsilon$
  - <u>Not</u> meaningful for $L \leq 0$, irrespective of ε
- **<u>RO Heuristic</u>**: $\varepsilon' \leq \varepsilon + \varepsilon \cdot 2^{-L}$
  - Suffices to have $L \geq 0$ (no entropy loss) for $\varepsilon' = 2\varepsilon$
  - Meaningful for $L \leq 0$, "borrow" security from application
- **<u>Our Result</u>**: $\varepsilon' \leq \varepsilon + \sqrt{\varepsilon \cdot 2^{-L}}$
  - "Halfway in between" standard LHL and RO
    - Suffices to have $L \geq \log(1/\varepsilon)$ for $\varepsilon' = 2\varepsilon$
    - Like RO, <u>meaningful</u> for $L \leq 0$ (e.g. get $\varepsilon' = \sqrt{\varepsilon}$ when $L = 0$)

Which Applications?

- All "unpredictability" applications
  - MAC, signature, one-way-function, ID scheme, …
- Prominent "indistinguishability" applications
  - (stateless) CPA/CCA secure encryption, weak PRFs
  - But not PRFs, PRPs, stream ciphers, one-time pad
    - Note: OK to derive AES key for CPA encryption/MAC !
- Observation: composing with a weak PRF, can include any (computationally-secure) application !
  - E.g., PRFs/PRPs/stream ciphers, but not one-time pad
  - Cost: one wPRF call + wPRF input now part of the seed

- Part II: Improving the Seed Length

Expand-then-Extract

- Recall, best $n = O(\text{sec. param. } k)$
  - But LHL needs $n \geq |X|$
- <u>Idea</u>: use pseudorandom generator (PRG) G to expand the seed from k bits to $n = |X|$ bits:

  Ext'(X; s) = Ext(X; G(s))

  - Friendly to "streaming" sources
  - Can result in <u>very fast</u> implementations

- <u>Hope</u>: extracted bits are pseudorandom
- Is this idea sound?

Soundness of Expand-then-Extract

- Trivial: $(\mathbf{Ext}(X; G(S)), G(S)) \approx_c (U_v, G(S))$
  - Otherwise distinguish $G(U_k)$ from U_n
- Problem: need $(\mathbf{Ext}(X; G(S)), S) \approx_c (U_v, S)$ (*)
- Theorem 1: Under DDH assumption, there exists a PRG G and a universal hash function Ext (thus, extractor, by LHL) s.t. can break (*) efficiently with advantage ≈ 1 on any source X
  - Thus, expand-then-extract might be insecure ☹

OK to Extract Small Number of Bits!

- Theorem 2: Extract-then-expand is secure when number of extracted bits v < "log(PRG security)"
  - Note 1: PRG should be secure against $O(\frac{\exp(v)}{\varepsilon})$ size circuits
  - Note 2: extracted bits are still statistically random !
  - Note 3: same min-entropy m, error drops to $\sqrt{\varepsilon}$
- Corollary: always safe to extract $v = O(\log k)$ bits, sometimes might be safe to extract $v = \Omega(k)$ bits ☺
- Seed Length n ? At best, $n = O(v + \log(1/\varepsilon))$, same as "almost universal" hash functions ☹

Expand-then-Extract Secure in **Minicrypt**

- Counter-example used DDH – "public-key gadget"
- **Minicrypt**: one of Impagliazzo's worlds, where PRGs exist but no public-key encryption (PKE)
- Theorem 3: Extract-then-expand is secure in **Minicrypt**
  - True for any number of extracted bits, but "settle" for efficiently samplable sources and pseudorandom bits
  - Similar in spirit to [HN, Pie, Dzi, DI, PS], but simpler!

Expand-then-Extract Secure in **Minicrypt**

- Theorem 3: if X is efficiently samplable, G is a PRG and D efficiently distinguishes $(\text{Ext}(X; G(S)), S)$ from (U, S), then PKE exist
- Secret Key $= S$, Public Key $= G(S)$
- Encryption $\text{Enc}_{PK}(b)$: send ciphertext R, where
  - if $b = 0$, sample X and set $R \leftarrow \text{Ext}(X; G(S))$
  - if $b = 1$, set $R \leftarrow U$
- Decryption $\text{Dec}_{SK}(R)$: use $D(R, S)$ to recover b
- Semantic security follows from PRG security:

$$(\text{Ext}(X; G(S)), G(S)) \approx_c (U, G(S)) \quad \blacksquare$$

Interpretation

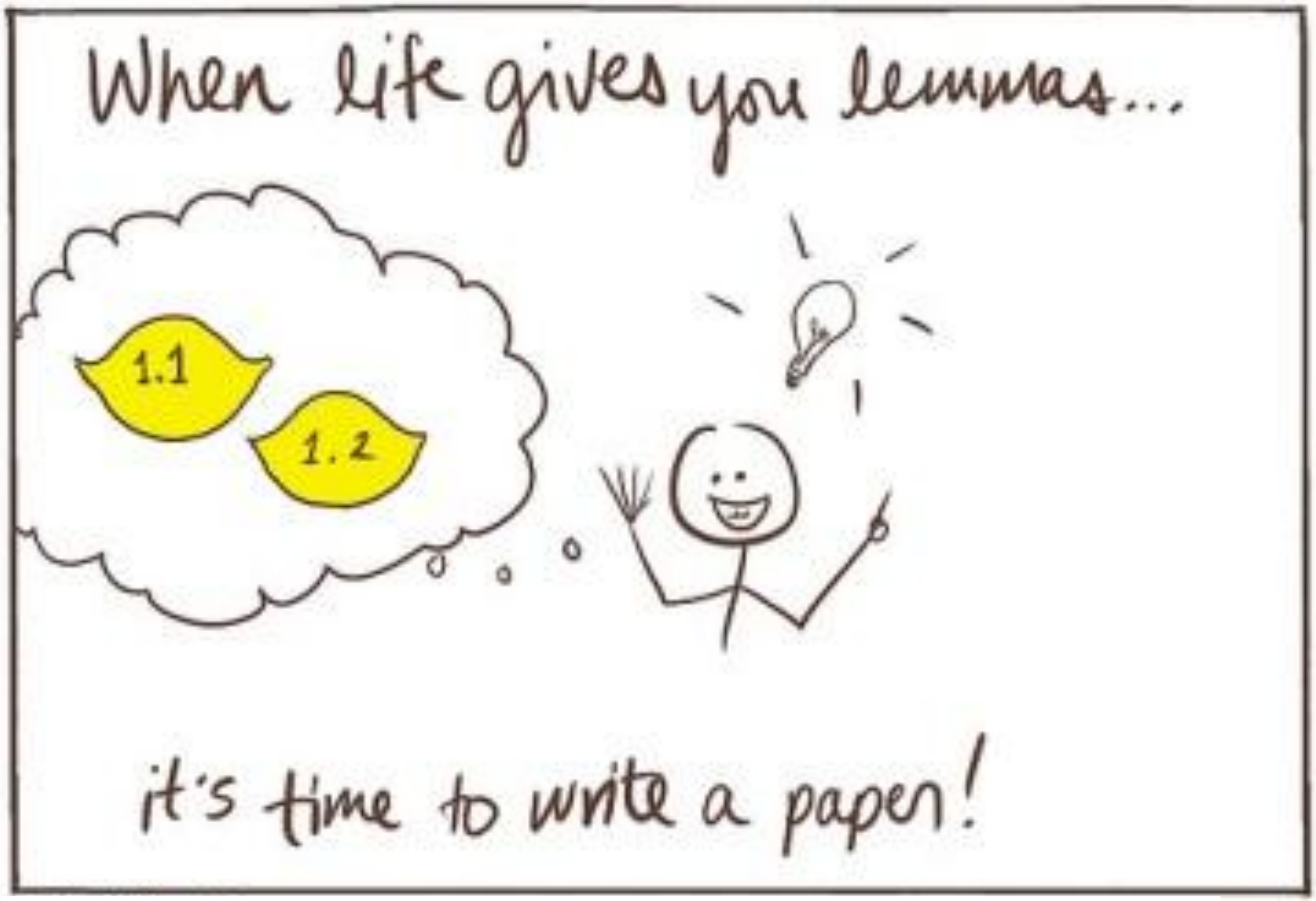
"It's not what it looks like"

Interpretation

- **<u>Corollary</u>**: Let G be a PRG.
  Assume there exists no PKE with sk = S, pk = G(S), pseudorandom ciphertexts and ≈ same security as G.
  Then expand-then-extract is secure with G.
- "Practical" PRGs (e.g. AES) unlikely to yield such a PKE
  - No black-box construction known (even with powerful "cryptomania" assumptions, like NIZK, IBE, FHE, etc.)
  - Possible that no PKE is as secure as AES !
  - Would be a major breakthrough with, say, AES
- <u>Moral</u>: formal evidence that expand-then-extract might be "secure in practice" (with "actually used" ciphers)

Summary

- Can improve large *entropy loss* and *seed length* of LHL
- Entropy loss: for a wide range of applications reduce entropy loss from $2\log(1/\varepsilon)$ to $\log(1/\varepsilon)$
  - Directly includes all authentication and some privacy applications (including CPA encryption, weak PRFs)
  - Using wPRFs, computational extractor for all applications!
- Seed length: expand-then-extract approach
  - Not sound in general…
  - Sound for extracting small # of bits
  - Sound for "practical" PRGs (which do not "imply" PKE)



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