Fast Cryptographic Primitives & Circular-Secure Encryption Based on Hard Learning Problems

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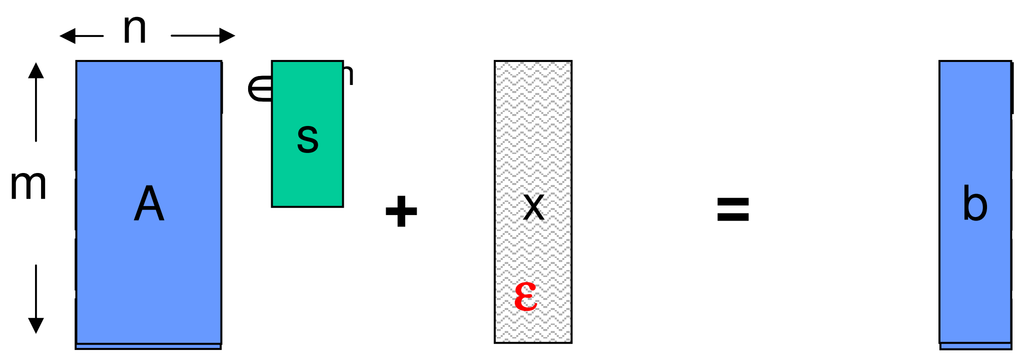
Learning Noisy Linear Functions

Learning Parity with Noise (LPN)

Problem: find s

$\mathbf{s} \in Z_2^n$

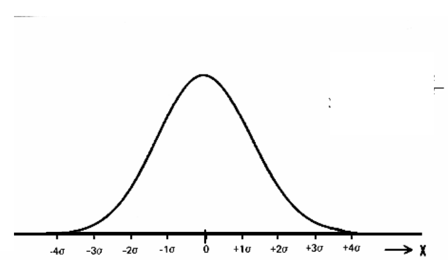


$=\langle a_i, s\rangle + \text{noise}$

iid noise vector of rate ε

e.g., $\varepsilon = 1/4$

- Extension to larger moduli: Learning-with-Errors (LWE) [Reg05] :
  - Z_q where $q(n)=\text{poly}(n)$ is typically prime
  - Gaussian noise w/mean 0 and std $\approx$ sqrt(q)



-(q-1)/2 0 (q-1)/2

Learning Noisy Linear Functions

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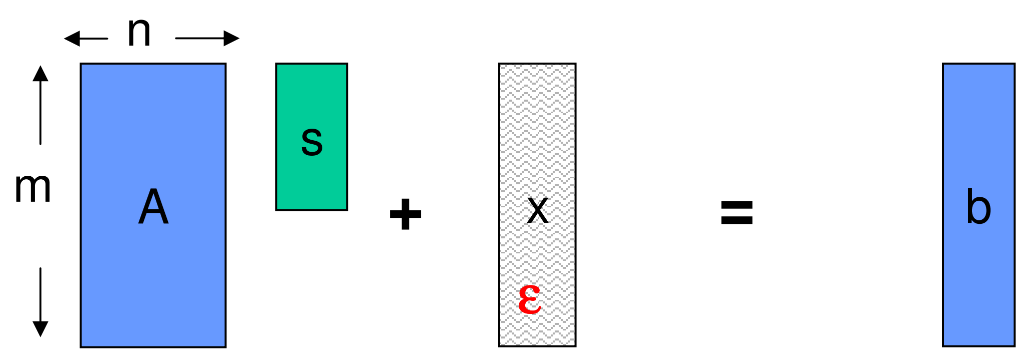


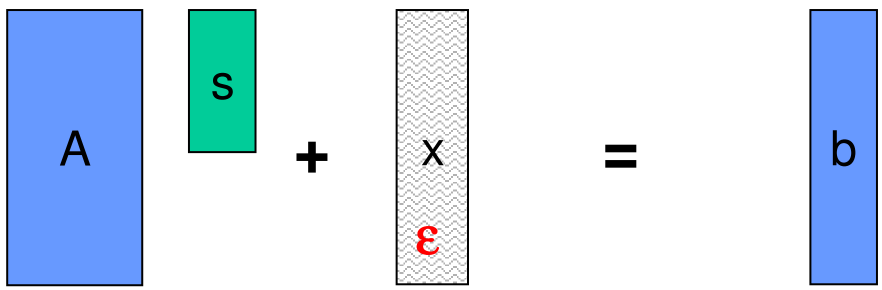
- Assumption: LWE/LPN is computationally hard for all m=poly(n)
- Well studied in Coding Theory/Learning Theory/ Crypto [GKL93,BFKL93, Chab94,Kearns98,BKW00,HB01,JW05,Lyu05,FGKP06,KS06,PW08,GPV08,PVW08…]
- Pros:
  - Reduction from worst-case Lattice problems [Reg05,Peik09]
  - Hardness of search problem
  - So far resists sub-exp & quantum attacks

Why LWE/LPN ?

- Problem has simple algebraic structure: “almost linear” function
  - exploited by [BFKL94, AIK07, D-TK-L09]
- Computable by simple (bit) operations (low hardware complexity)
  - exploited by [HB01,AIK04,JW05]

rare combination

- Message of this talk: Very useful combination



Main Results

This talk:

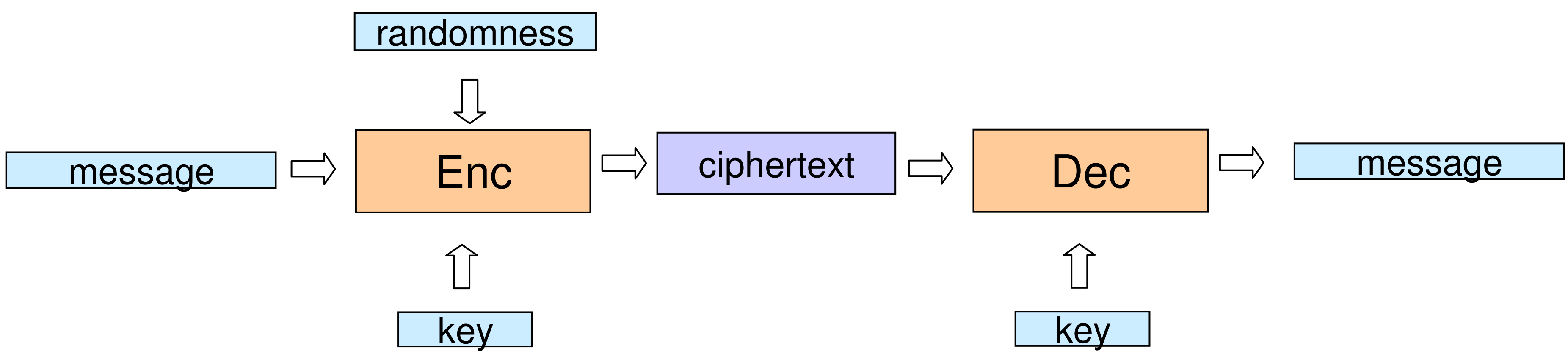
- Fast circular secure encryption schemes
  - Symmetric encryption from LPN
  - Public-key encryption from LWE

- Fast pseudorandom objects from LPN
  - Pseudorandom generator $G:\{0,1\}^n \to \{0,1\}^{2n}$ in quasi-linear time
  - Oblivious weak randomized pseudorandom function

Encryption Scheme

- Security: Even if **Adv** gets information cannot break scheme.
  - CPA [GM82]:given oracle to $E_{key}()$ can't distinguish $E_k(m_1)$ from $E_k(m_2)$

- What if **Adv** sees $E_k(msg)$ where msg depends on the key (KDM attack)?
  - E.g., $E_{key}(key)$ or $E_{key}(f(key))$ or $E_{k1}(k_2)$ and $E_{k2}(k_1)$



KDM / circular security

F-KDM Security [BlackRogawayShrimpton02] : Adv gets $E_k(f(k))$ for $f \in F$

Circular security [CamenischLysyanskaya01] : Adv gets $E_{k1}(k_2)$, $E_{k2}(k_3)\ldots$, $E_{ki}(k_1)$

Can we achieve KDM/circular security?

- many recent works [BRS02, HK07, BPS07, BHHO08, CCS08, BDU08, HU08,HH08]
- natural questi
  - disk encrypti
  - anonymous 01]
  - axiomatic se
  - Gentry's fully homomorphic scheme [Gen09]

[BHHO08]: **Yes, we can !**

- non-trivial to achieve:
  - some ciphers become insecure under KDM attacks (e.g.,AES in LRW mode)
  - random oracle constructions are problematic [HofheintzUnruh08,HaleviKrawczyk07]
  - can't get KDM from trapdoor permutation in a black-box way [HaitnerHolenstein08]

BHHO Scheme vs. Our Scheme

- [BonehHaleviHamburgOstrovsky08] First circular public-key scheme from DDH
  - Get "clique" security + KDM for affine functions
  - But large computational/communication overhead
  - t-bit message: **Time**: t exponentiations (compare to El-Gamal)

    Communication: t group elements

- Our schemes: circular encryption under LPN/LWE
  - Get "clique" security + KDM for affine functions
  - Proofs of security follow the [BHHO08] approach
  - Circular security comes "for free" from standard schemes
  - Efficiency comparable to standard LWE/LPN schemes
  - t-bit message: **Time**: symmetric case: $t \cdot \text{polylog}(t)$;

    public-key: $t^2 \cdot \text{polylog}(t)$

    Communication: $O(t)$ bits.

Symmetric Scheme from LPN

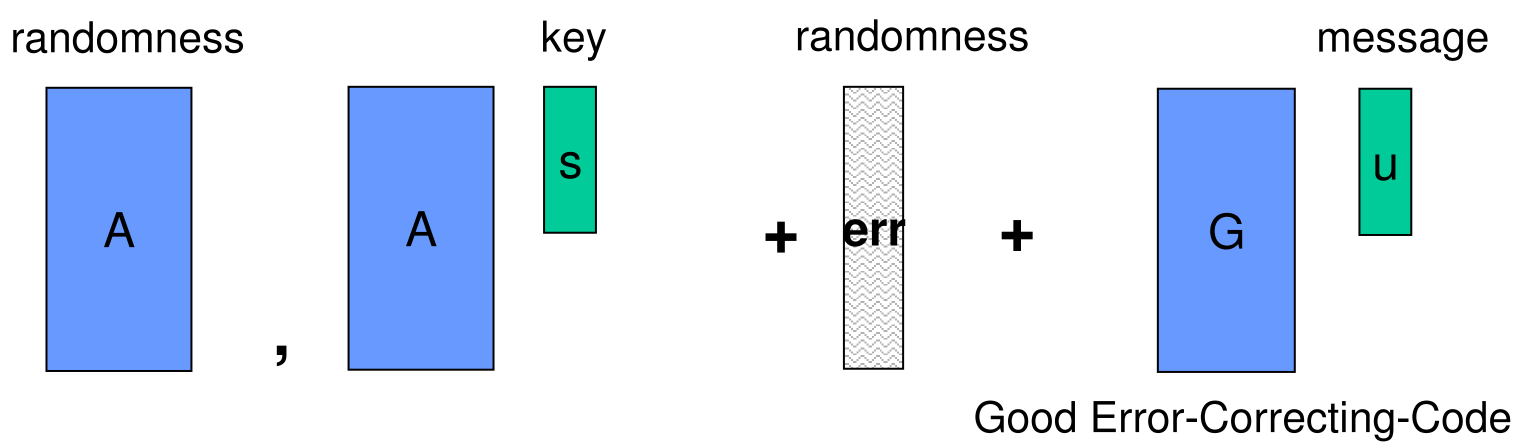
Symmetric Scheme

- Let G be a good linear error-correcting code with decoder for noise ε+0.1

$\mathrm{Enc}_s(\mathrm{mes};\, A,\, \mathrm{err}) = (A,\, As + \mathrm{err} + G \cdot \mathrm{mes})$

$\mathrm{Dec}_s(A, y) = \mathrm{decoder}(y - As)$

- Natural scheme originally from [GilbertRobshawSeurin08]
  - independently discovered by [A08,DodisTauman-KalaiLovet09]
- Also obtain amortized version with quasilinear implementation (See paper)



Clique Security

$Enc_s(mes; A, err) = (A, As+err + G\cdot mes)$

$Dec_s(A,y) = decoder(y-As)$

Thm. Scheme is circular (clique) secure and KDM w/r to affine functions

Proof:

- Useful properties:
  - Plaintext homomorphic: Given $E_s(u)$ and v can compute $E_s(u+v)$

$$(A, As+err + G\cdot(u+v) + G\cdot v \quad)$$

Clique Security

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Proof:

- Useful properties:
  - Plaintext homomorphic: Given $E_s(u)$ and v can compute $E_s(v+u)$
  - Key homomorphic: Given $E_s(u)$ and r can compute $E_{s+r}(u)$

$$(A, A\cdot(s+r)+err+Gu \; +A\cdot r \;)$$

Clique Security

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  - Key homomorphic: Given $E_s(u)$ and r can compute $E_{s+r}(u)$
  - Self referential: Given $E_s(0)$ can compute $E_s(s)$

$$(A - G, \; As + err)$$

$$= (A', \; (A'+G)s + err)$$

$$= (A', \; A's + err + Gs)$$

$$= E_s(s)$$

Clique Security

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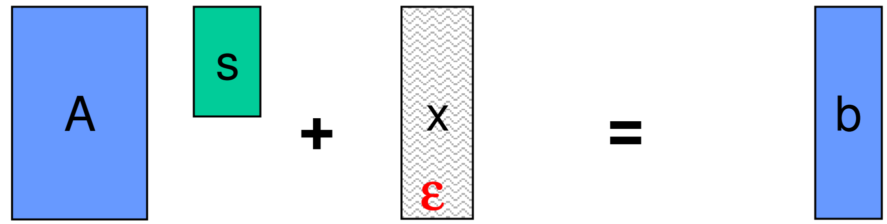
  - Key homomorphic: Given $E_s(u)$ and r can compute $E_{s+r}(u)$
  - Self referential: Given $E_s(0)$ can compute $E_s(s)$
- Suppose that Adv break clique security (can ask for $E_{Si}(S_k)$ for all $1 \leq i,k \leq t$)
- Construct B that breaks standard CPA security (w/r to single key S).
- B simulates Adv: choose t offsets $\Delta_1,\ldots, \Delta_t$ and pretend that $S_i=S+\Delta_i$
  - Simulate $E_{si}(S_k)$: get $E_s(0) \rightarrow E_s(S) \rightarrow E_{s+\Delta i}(S) \rightarrow E_{s+\Delta i}(S+\Delta_k)$

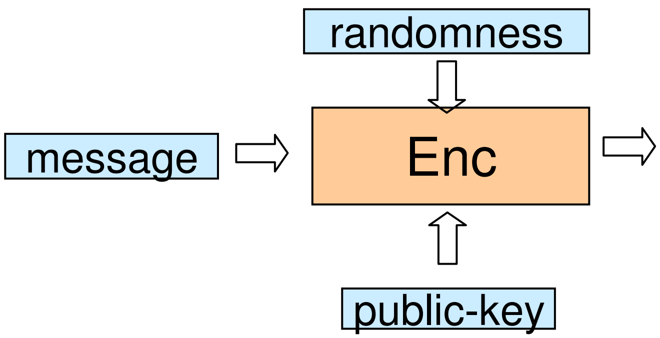
Public-key Scheme from LWE

Regev's Scheme - [GPV-PVW08] variant

- Public-key: $A \in Z_q^{n\times m}$, $b \in Z_q^m$
- Secret-key: $s \in Z_q^n$



- Encrypt $z \in Z_p \subset Z_q$ by $(u \in Z_q^n, c \in Z_q)$



random vector

fixed linear ECC

$(u, \langle s,u\rangle + err + g\cdot(message))$

distribution over low-weight elements

- To Decrypt (u,c): compute $c - \langle s,u\rangle = g\cdot mes + err$ and decode
- CPA Security in [Regev05, GentryPeikertVaikuntanathan08]
- Want: Plaintext homomorphic, Self referential, Key homomorphic

Regev's Scheme - [GPV-PVW08] variant

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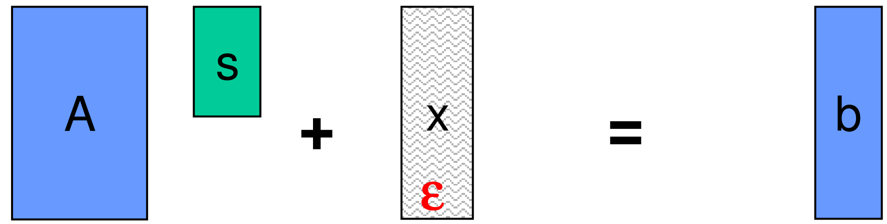

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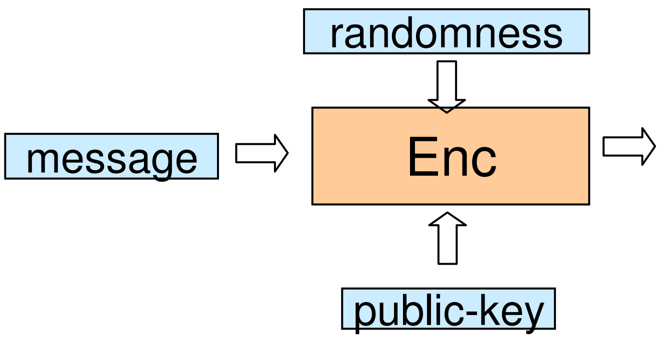


random vector

fixed linear ECC

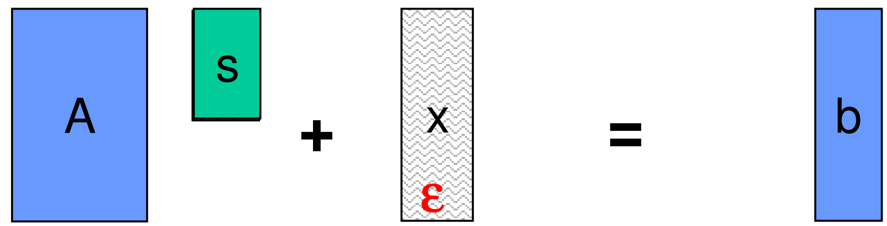
$(u, \langle s,u\rangle + \text{err} + g\cdot(\text{message}))$

distribution over low-weight elements

- To Decrypt (u,c): compute $c - \langle s,u\rangle = g\cdot \text{mes} + \text{err}$ and decode
- CPA Security in [Regev05, GentryPeikertVaikuntanathan08]
- Want: Plaintext homomorphic, **Self referential**, Key homomorphic

Self Reference

- Public-key: $A \in Z_q^{n \times m}$, $b \in Z_q^m$
- Secret-key: $s \in Z_q^n$



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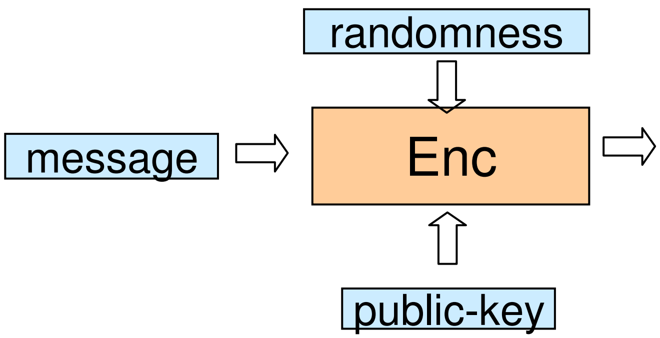


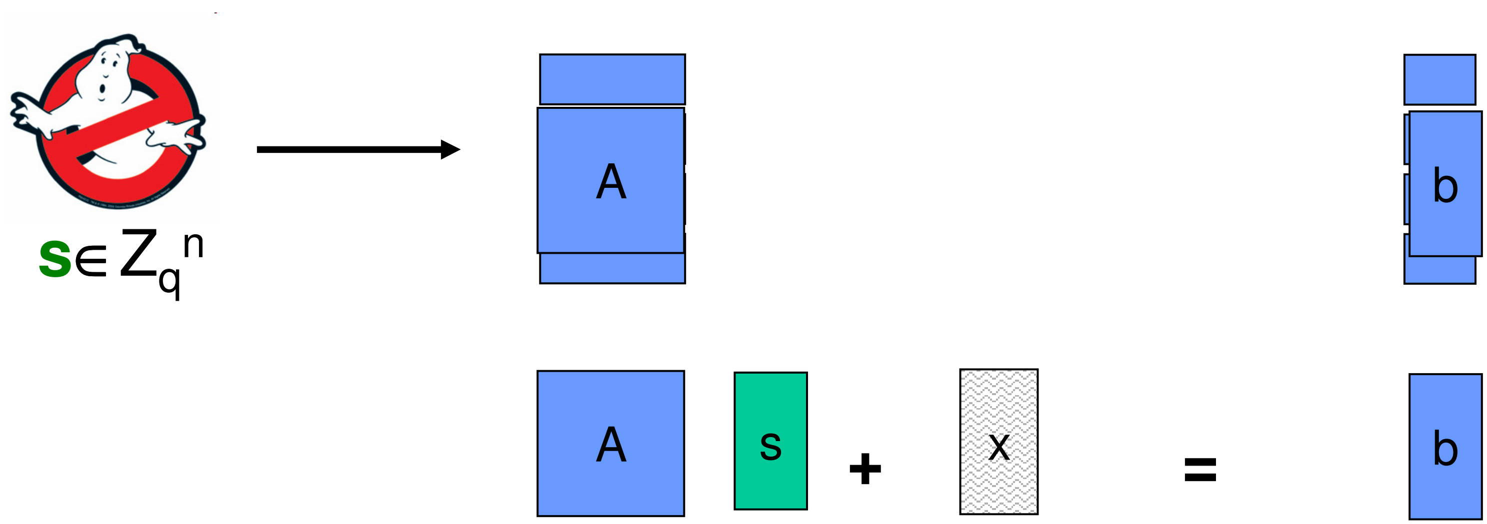
$$(u, \langle s,u \rangle + err + g \cdot (\text{message}))$$

- Can we convert E(0) to E(s_1) ?
- Can use prev ideas (up to some technicalities) but…
- Problem: s_1 may not be in Z_p
- Sol: Choose s with entries in Z_p by sampling from Gaussian around $(0 \pm p/2)$
- Security: we show how to convert standard LWE to LWE with $s \leftarrow$ Noise

Hardness of LWE with s←Noise

Convert standard LWE to LWE with s←Noise

1. Get (A,b) s.t A is invertible



Hardness of LWE with s←Noise

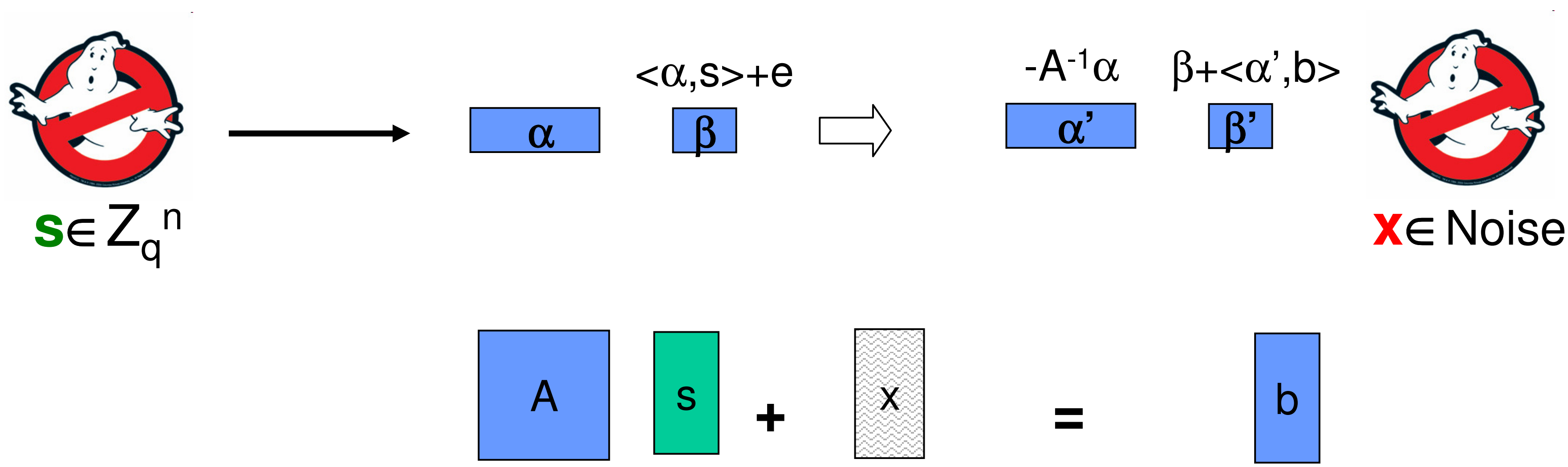
Convert standard LWE to LWE with s←Noise

- If $(\alpha,\beta) \leftarrow LWE_s$ then $(\alpha',\beta') \leftarrow LWE_x$

  Proof: $\beta' = \beta + \langle \alpha', b \rangle$

  $= \langle \alpha, s \rangle + e + \langle \alpha', As \rangle + \langle \alpha', x \rangle$

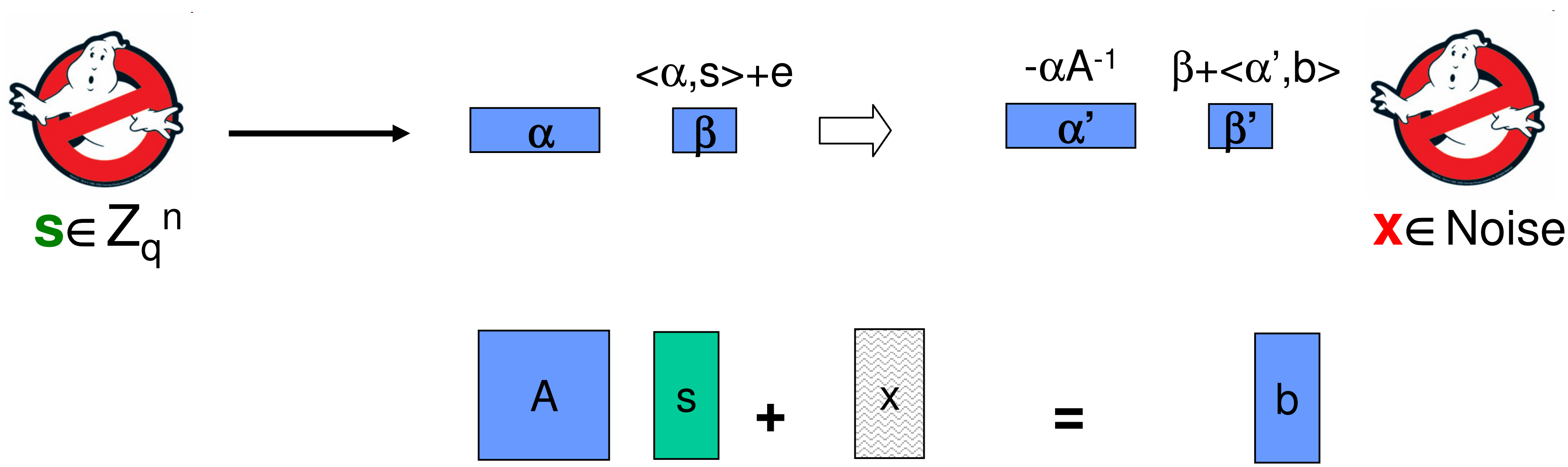
  $= \langle \cancel{\alpha, s} \rangle + e + \langle \cancel{-A^{-1}\alpha, A}s \rangle + \langle \alpha', x \rangle$



Hardness of LWE with $s \leftarrow$Noise

Convert standard LWE to LWE with $s \leftarrow$Noise

- If $(\alpha,\beta) \leftarrow LWE_s$ then $(\alpha',\beta') \leftarrow LWE_x$
- If (α,β) are uniform then (α',β') also uniform
- Hence distinguisher for LWE_x yields a distinguisher for LWE_s



Hardness of LWE with s←Noise

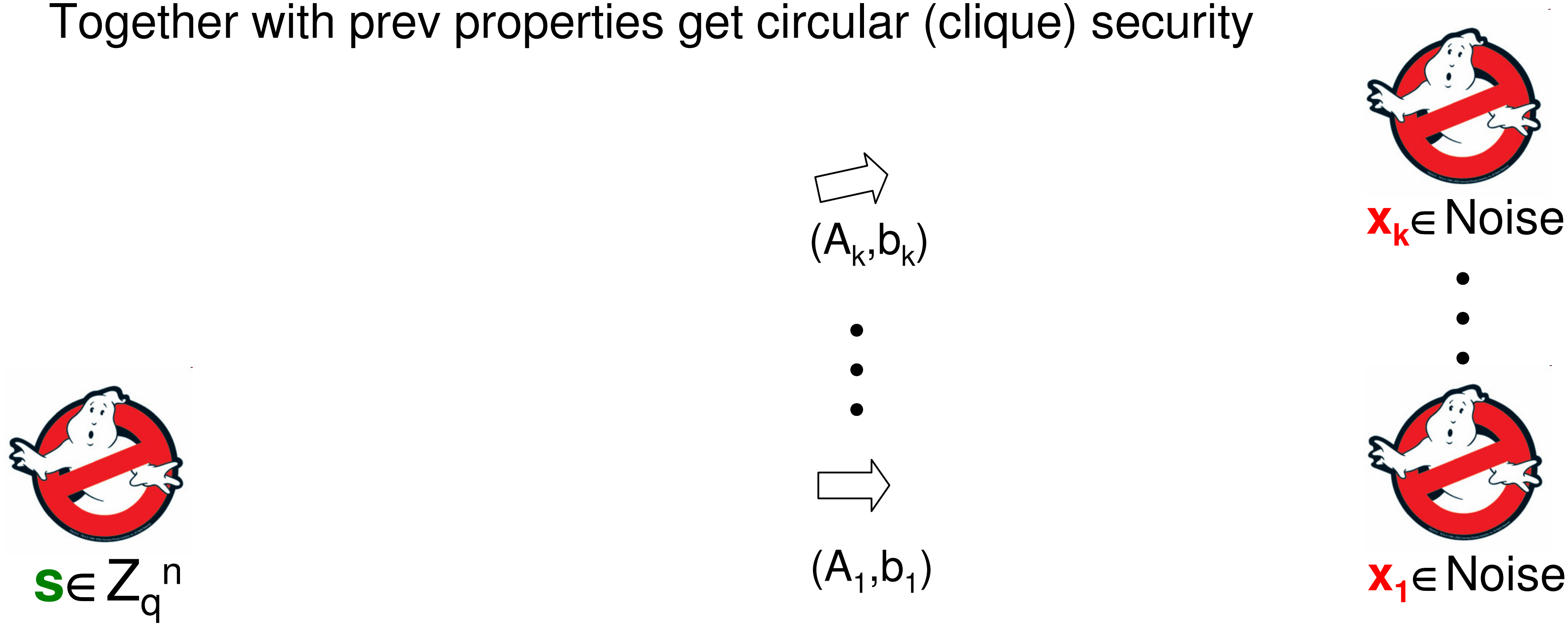
- Reduction generates invertible linear mapping $f_{A,b}$:s $\rightarrow$ x

s$\in Z_q^n$ $\Rightarrow$ (A,b) x$\in$ Noise

A s + x = b

Hardness of LWE with s←Noise

- Reduction generates invertible linear mapping $f_{A,b}$:s → x
- Key Hom: get pk's whose sk's $x_1,..,x_k$ satisfy known linear-relation
- Together with prev properties get circular (clique) security



- Improve efficiency via amortized version of [PVW08]

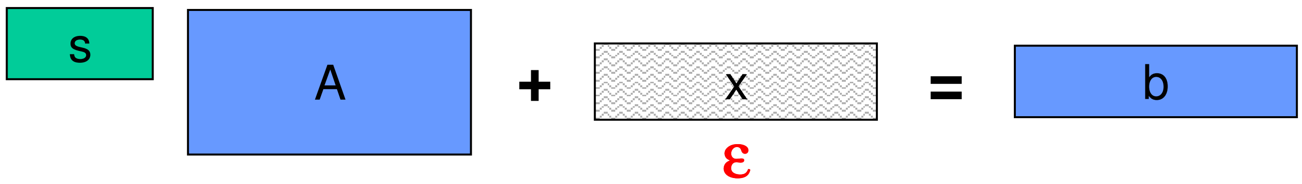
Open Questions

- LWE vs. LPN ?
  - LWE follows from worst-case lattice assumptions [Regev05, Peikert09]
  - LWE many important crypto applications [GPV08,PVW08,PW08,CPS09]
  - LWE can be broken in "NP$\cap$ co-NP" unknown for LPN
  - LPN central in learning ("complete" for learning via Fourier) [FeldmanGopalanKhotPonnuswami06]

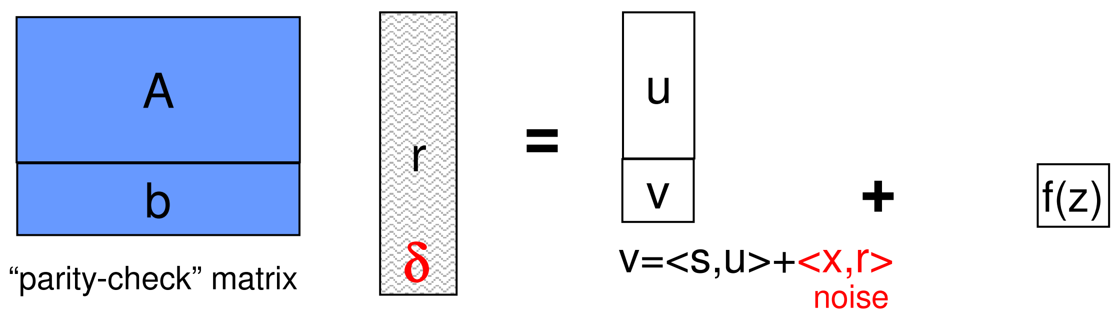
- Circular Security vs. Leakage Resistance ?
  - Current constructions coincident
  - LPN/Regev/BHHO constructions resist key-leakage [AkaviaGoldwasserVaikuntanathan09, DodisKalaiLovett09, NaorSegev09]
  - common natural ancestor?

Regev's Scheme - [GPV-PVW08] variant

- Public-key: $(A,b) \in Z_q^{n\times m} \times Z_q^m$ Secret-key: $s \in Z_q^n$

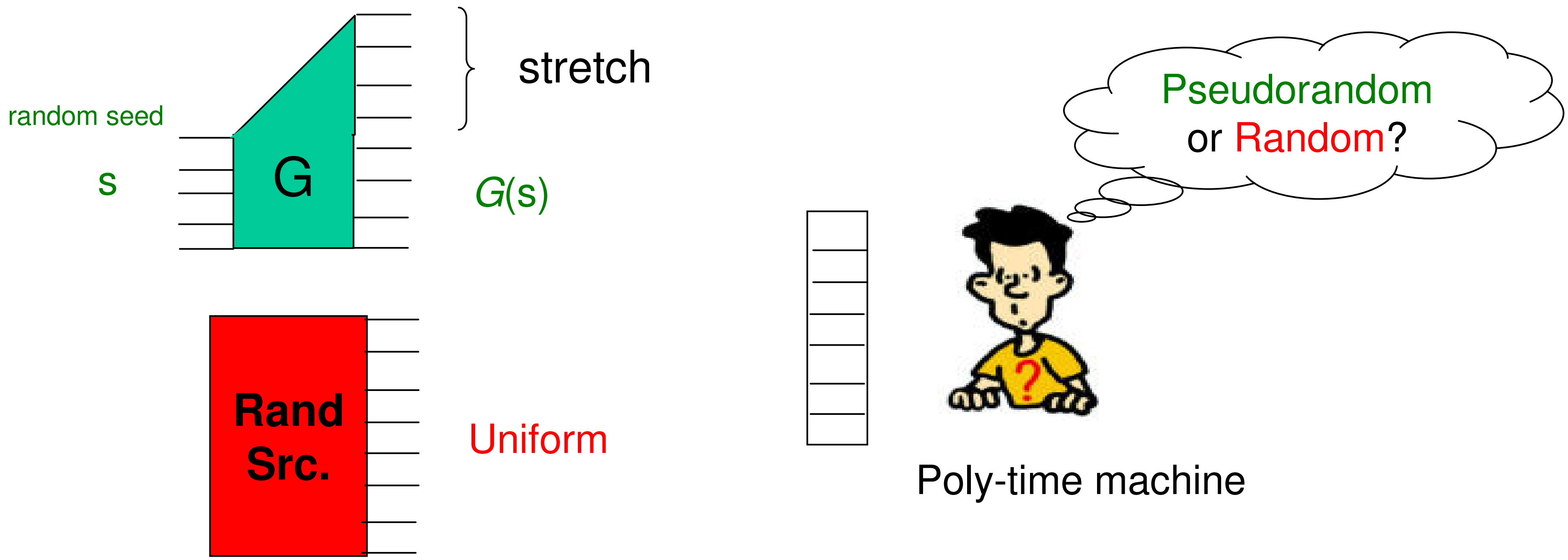


- Encrypt $z \in Z_p \subset Z_q$ by $(u,v+f(z))$ where $f: Z_p \to Z_q$ is linear ECC, i.e., $f(z)=az$



- To Decrypt (u,c): compute $c-\langle s,u\rangle=f(z)+\langle x,r\rangle$ and decode
- Security [R05,GPV]: If b was truly random then (u,v) is random and get OTP
- Want: Plaintext homomorphic, Self referential, Key homomorphic
- Plaintext hom: let message space be subgroup of Z_q by taking $q=p^2$

Pseudorandom Generator (PRG)



- Can be constructed from any one-way function [HILL90]
- Stretch of 1 bit $\Rightarrow$ Stretch of polynomially many bits [BM-Y, GM84]

Circuit Complexity of PRGs

Pseudorandom generator $G:\{0,1\}^n \rightarrow \{0,1\}^{2n}$

- At least $\Omega(n)$ circuit size
- Can we get low overhead of $O(n)$ or $n \cdot \text{polylog}(n)$?
  - natural question
  - [IKOS08] PRG with low overhead $\Rightarrow$ low-overhead cryptography
    e.g., PK-encryption in time O(|message|), for sufficiently large message.

Construction	Assumption	Time (circuit size)
[BM84, GM84]	1-bit PRG G'	$n \cdot \text{Time}(G') > n^2$
[Gen00,DRV02, DN02]	Number Theoretic	More than n^2
[BFKL94, FS96]	LPN	n^2
[AIK06]	sparse-LPN (non-standard)	n
This work	LPN (standard)	$n \cdot \text{polylog}(n)$

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[Genarro00, DedicReyzinVadhan02, DamgardNielsen02]	Number Theoretic	More than n^2
[BlumFurstKearnsLipton94, FischerStern96]	LPN	n^2
[A-IshaiKushilevitz06]	sparse-LPN (non-standard)	n
This work	LPN (standard)	$n \cdot \mathrm{polylog}(n)$

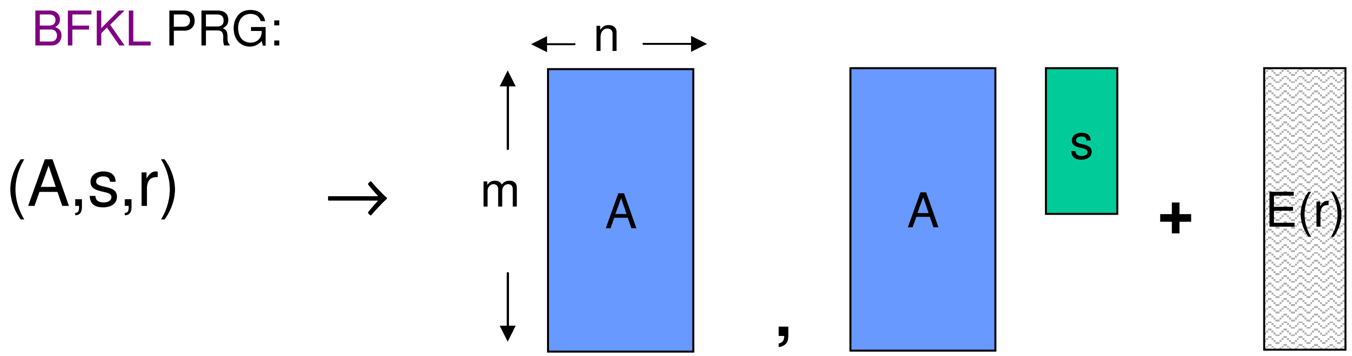
The [BFKL] generator

BFKL generator: $G(A, s, r) = (A, As + Err(r))$

- input: $nm+n+mH_2(\varepsilon)$ output: $nm+m$ stretch: $m(1-n/m - H_2(\varepsilon))$
- Efficiency: only bit operations !
- Bottleneck 1: at least $\Omega(mn)$ due to matrix-vector multiplication
- Bottleneck 2: Sampling Err(r) (with low randomness complexity) takes time

[FischerStern96] : quadratic time on a RAM machine

BFKL PRG:



Solving 1: Amortization

BFKL generator: G(A, s, r)= (A,As+ Err(r))

- Bottleneck 1: at least $\Omega(mn)$ due to matrix-vector multiplication
- Sol: Amortization
- Use many different s's with the same A
- Preserves pseudorandomness since A is public

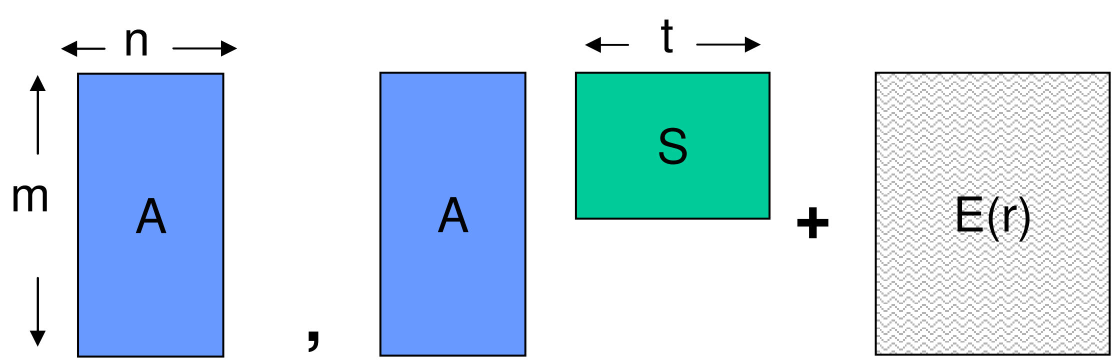
    -Proof via Hybrid argument

- If matrices are very rectangular can multiply in quasi-linear time [Cop82]

    - E.g., t=n and m=n^6

PRG:

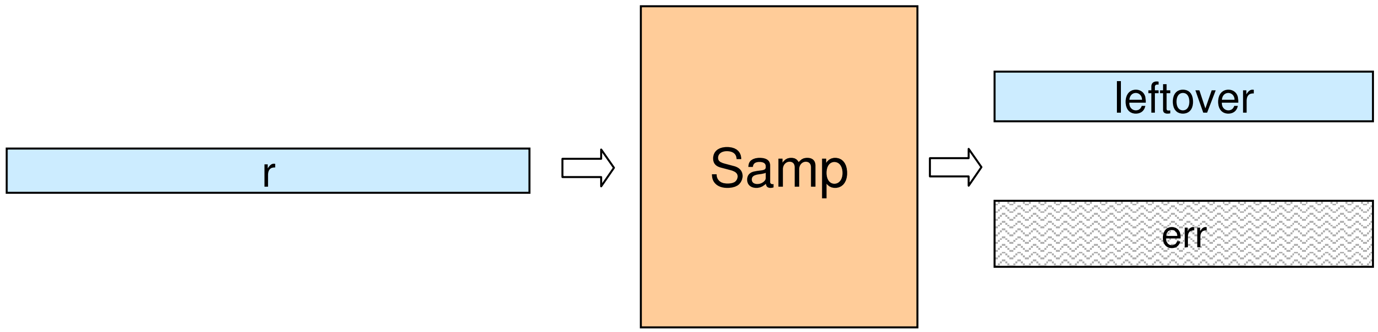
(A,S,r) $\rightarrow$



Solving 2: Sampling with leftovers

Bottleneck 2: Sampling noise w/low randomness takes $O(n^2)$

- Sol: [AIK06] Samp(r)= (err, leftover)



- PRG G(A,S,r)= (A, AS+err, leftover)
- How to sample w/leftovers?
  - If $\varepsilon=1/4$ partition r to pairs and let $err_i = r_{2i-1} \cdot r_{2i}$
  - r has a lot of entropy given err, so can extract the leftover
  - Can get linear time with leftover of linear length
- G has linear stretch and computable in quasi-linear time

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Conclusion and Open Questions

- DRLC is useful for private-key primitives that need
  - fast hardware implementation
  - special homomorphic properties

- Find more crypto application for DRLC
  - collision resistance hash-functions
  - public-key crypto [Alekh03] uses $m=O(n)$, $\varepsilon=\mathrm{sqrt}(n)$