Standard Security Does Imply Security Against Selective Opening for Markov Distributions

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Abstract. About three decades ago it was realized that implementing private channels between parties which can be adaptively corrupted requires an encryption scheme that is secure against *selective opening attacks*. Whether standard (IND-CPA) security implies security against selective opening attacks has been a major open question since. The only known reduction from selective opening to IND-CPA security loses an exponential factor. A polynomial reduction is only known for the very special case where the distribution considered in the selective opening security experiment is a product distribution, i.e., the messages are sampled independently from each other.
In this paper we give a reduction whose loss is quantified via the dependence graph (where message dependencies correspond to edges) of the underlying message distribution. In particular, for some concrete distributions including Markov distributions, our reduction is polynomial.

Keywords. Public-key encryption, selective opening security, Markov, IND-CPA, IND-SO-CPA

1 Introduction

Security under Selective Opening Attacks. Consider a scenario where many parties $1, \ldots, n$ send messages to one common receiver. To transmit a message $\mathbf{m}_i$, party i samples fresh randomness $\mathbf{r}_i$ and sends the ciphertext $\mathbf{c}_i = \mathsf{Enc}_{pk}(\mathbf{m}_i; \mathbf{r}_i)$ to the receiver. Consider an adversary $\mathcal{A}$ that does not only eavesdrop on the sent ciphertexts $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$, but corrupts a set $\mathcal{I} \subseteq [n]$ of the sender's systems, thus learning the encrypted message $\mathbf{m}_i$ *and* the randomness $\mathbf{r}_i$ used to encrypt $\mathbf{m}_i$. The natural question to ask is whether the messages of uncorrupted parties remain confidential. Such attacks are referred to as *selective opening* (SO) *attacks* (under sender corruption).

Selective opening attacks naturally occur in multi-party computation where we assume secure channels between parties. Since a party might become corrupted, we would need the encryption on the channels to be selective opening secure. In practice the same argument applies to a server that establishes secure connections that shall remain secure if users are corrupted.

Difficulty of Proving Security under Selective Opening Attacks. The widely accepted standard notion for public-key encryption schemes is indistinguishability under chosen-plaintext attacks (IND-CPA security). At first sight one might consider a straight-forward hybrid argument to show that IND-CPA security already implies security against selective opening attacks since every party samples fresh randomness independently. However, so far nobody has been able to bring forward such a hybrid argument in general. Notice that revealing randomness $\mathbf{r}_i$ allows a selective opening adversary to verify that a corrupted ciphertext $\mathbf{c}_i$ is an encryption of $\mathbf{m}_i$. The adversary's possibility to corrupt parties introduces a difficulty in proving that standard (IND-CPA) security already implies selective opening security. It seems that the reduction has to know (i.e. guess) the complete set $\mathcal{I}$ of all corruptions going to be made by $\mathcal{A}$ in order to serve its security game *before* $\mathcal{A}$ actually announces the senders it wishes to corrupt. Since $\mathcal{I}$ might be any subset of $\{1, \ldots, n\}$, a direct approach would lead to an exponential loss in the reduction. A main technical obstacle is that the encrypted messages may depend on each other. If, for example, they are encrypted and sent sequentially, message $\mathbf{m}_i$ may depend on $\mathbf{m}_{i-1}$ and all previous messages. Thus, corrupting some parties might already leak some information on messages sent by parties that have not been corrupted.

Until today, the only result in the standard model, given in [8, 3], shows that IND-CPA implies selective opening security for the special case of a product distribution, i.e., when all messages $\mathbf{m}_1, \ldots, \mathbf{m}_n$ are sampled independently from each other. Intuitively, this holds since corrupting some ciphertext cannot reveal information on related messages if there are no related messages at all and the hybrid argument one might expect to work goes through. This leaves the following open question:

Does standard security imply selective opening security for any non-trivial message distribution?

1.1 Our Contributions

We present the first non-trivial positive results in the standard model, namely we show that IND-CPA security implies IND-SO-CPA security for a class of message distributions with few dependencies. Here IND-SO-CPA security refers to the indistinguishability-based definition of selective opening security sometimes referred to as *weak* IND-SO-CPA *security* [4].

IND-SO-CPA requires that a passive adversary that obtains a vector of ciphertexts $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ and has access to a ciphertext *opening* oracle, revealing the underlying message $\mathbf{m}_i$ of some ciphertext $\mathbf{c}_i$ and the randomness used to encrypt $\mathbf{m}_i$, cannot distinguish the originally encrypted messages from freshly resampled messages that are *as likely as the original messages* given the messages of opened ciphertexts.

We consider *graph-induced* distributions where dependencies among messages correspond to edges in a graph and show that IND-CPA implies IND-SO-CPA

security for all graph-induced distributions that satisfy a certain *low connectivity* property.

In particular, our result holds for the class of Markov distributions, i.e. distributions on message vectors $(\mathbf{m}_1, \ldots, \mathbf{m}_n)$ where all information relevant for the distribution of $\mathbf{m}_i$ is present in $\mathbf{m}_{i-1}$. We prove that any IND-CPA secure public-key encryption scheme is IND-SO-CPA secure if the messages are sampled from a Markov distribution. Our results cover for instance distributions where message $\mathbf{m}_i$ contains all previous messages (e.g. email conversations) or distributions where messages are increasing, i.e., $\mathbf{m}_1 \leq \mathbf{m}_2 \leq \ldots \leq \mathbf{m}_n$.

Note that a positive result on "weak" IND-SO-CPA security for all IND-CPA-secure encryption schemes for certain distributions is the best we can hope for due to the negative result of Bellare et al. [1] ruling out such an implication for SIM-SO-CPA security.

Details. Think of a vector of n messages sampled from some distribution $\mathfrak{D}$ as a graph G on n vertices $\{1, \ldots, n\}$ where we have an edge from message $\mathbf{m}_i$ to message $\mathbf{m}_j$ if the distribution of $\mathbf{m}_j$ depends on $\mathbf{m}_i$. Further, fix any subset $\mathcal{I} \subseteq \{1, \ldots, n\}$ of opening queries made by some adversary. The main observation is that removing $\mathcal{I}$ and all incident edges, G decomposes into connected components $C_1, \ldots, C_{n'}$ that can be resampled independently, since the distribution of messages on C_k solely depends on the messages in the neighborhood of C_k and $\mathfrak{D}$.

To argue that there is no efficient adversary $\mathcal{A}_{\mathsf{SO}}$ that distinguishes sampled and resampled messages in the selective opening experiment, we proceed in a sequence of hybrid games, starting in a game where after receiving encryptions of sampling messages and replies to opening queries, $\mathcal{A}_{\mathsf{SO}}$ obtains the sampled messages. In each hybrid step we use IND-CPA security to replace *sampled* messages on a connected component C_k with *resampled* messages without $\mathcal{A}_{\mathsf{SO}}$ noticing. To this end, the reduction from IND-CPA to the indistinguishability of two consecutive hybrids has to identify C_k to embed its own challenge before $\mathcal{A}_{\mathsf{SO}}$ makes any opening query.

We consider two approaches for guessing C_k. The first will consider graphs that have only polynomially many connected subgraphs; hence, the reduction can guess C_k right away. The second approach studies graphs for which every connected subgraph has a neighborhood of constant size; this allows the reduction to guess C_k by guessing its neighborhood. We show that the first approach ensures a reduction with polynomial loss for a strictly greater class of graphs than the second one.

Additionally, when the distribution is induced by an acyclic graph, we give a more sophisticated hybrid argument for the second approach, where in each hybrid transition only a single sampled message is replaced by a resampled message, allowing for a tighter reduction. Due to the definition of the hybrids, it will suffice to guess on fewer vertices of C_k's neighborhood.

1.2 Previous Work

There are three not polynomially equivalent definitions of SO-secure encryption [4]. Since messages in the IND-SO experiment have to be resampled conditioned on opened messages, there are two notions based on indistinguishability: *Weak* IND-SO restricts to distributions that support *efficient conditional* resampling. Bellare et al. [2] gave an indistinguishability-based notion for passive adversaries, usually referred to as IND-SO-CPA. *Full* IND-SO allows for arbitrary distributions on the messages and is due to Böhl et al. [4], who adopted a notion for commitment schemes from [2] to encryption.

SIM-SO captures semantic security and demands that everything an adversary can output can be computed by a simulator that only sees the messages of corrupted parties, whereas it does not see the public key, any ciphertext or any randomness. The notion dates back to Dwork et al. [8], who studied the *selective decommitment* problem, and does not suffer from a distribution restriction like *weak* IND-SO, since it does not involve resampling.

The first IND-SO-CPA-secure encryption scheme in the standard model was given in [2] based on lossy encryption. Selective opening secure encryption can be constructed from *deniable encryption* [6] as well as *non-committing encryption* [7]. Bellare et al. [3, 1] separated SIM-SO-CPA from IND-CPA security and showed that IND-CPA security implies *weak* IND-SO-CPA security if the messages are (basically) sampled independently. The same result was already established for commitment schemes in [8].

To date, this is the only positive result that shows that IND-CPA implies *weak* IND-SO-CPA in the standard model. *Full* IND-SO-CPA and SIM-SO-CPA security were separated in [4]; neither of them implies the other. Hofheinz et al. [10] proved that IND-CPA implies *weak* IND-SO-CPA in the generic group model for a certain class of encryption schemes and separated IND-CCA from *weak* IND-SO-CCA security.

Recently, Hofheinz et al. [9] constructed the first (even IND-CCA-secure) PKE that is not *weakly* IND-SO-CPA secure. Their result relies on the existence of *public-coin differing-inputs obfuscation* and certain *correlation intractable hash functions*. Their scheme employs "secret-sharing message distributions" whose messages are evaluations of some polynomial. It is easily seen that such distributions have too many dependencies to be covered by our positive result. There is a gap between their result and ours, that is, distributions for which it is still open whether IND-CPA implies IND-SO-CPA.

2 Preliminaries

We denote by λ the security parameter. A function f is polynomial in n, $f(n) = \mathsf{poly}(n)$, if $f(n) = \mathcal{O}(n^c)$ for some $c > 0$. Let $0 < n := n(\lambda) = \mathsf{poly}(\lambda)$. A function $f(n)$ is negligible in n, $f(n) = \mathsf{negl}(n)$, if $f(n) = \mathcal{O}(n^{-c})$ for all $c > 0$. Any algorithm receives the unary representation 1^λ of the security parameter as first input. We say that an algorithm is a PPT algorithm if it runs in probabilistic

polynomial time (in λ). For a finite set $\mathcal{S}$ we denote the sampling of a uniform random element a by $a \leftarrow\!\!\$\, \mathcal{S}$, and the sampling according to some distribution $\mathfrak{D}$ by $a \leftarrow \mathfrak{D}$. For $a, b \in \mathbb{N}$, $a \leq b$, let $[a, b] := \{a, a+1, \ldots, b\}$ and $[a] := [1, a]$. For $a < b$ let $[b, a] := \emptyset$. For $\mathcal{I} \subseteq [n]$ let $\overline{\mathcal{I}} := [n] \setminus \mathcal{I}$. We use boldface letters to denote vectors, which are of length n if not indicated otherwise. For a vector $\mathbf{m}$ and $i \in [n]$ let $\mathbf{m}_i$ denote the i-th entry of $\mathbf{m}$ and $|\mathbf{m}|$ the number of entries in $\mathbf{m}$. For a set $\mathcal{I} = \{i_1, \ldots, i_{|\mathcal{I}|}\}$, $i_1 < \ldots < i_{|\mathcal{I}|}$ let $\mathbf{m}_{\mathcal{I}}$ denote the projection of $\mathbf{m}$ to its $\mathcal{I}$-entries: $\mathbf{m}_{\mathcal{I}} := (\mathbf{m}_{i_1}, \ldots, \mathbf{m}_{i_{|\mathcal{I}|}})$. For an event $\mathtt{E}$ let $\overline{\mathtt{E}}$ denote the complementary event.

2.1 Games

A game G is a collection of procedures or oracles $\{\textsc{Initialize}, \mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_t, \textsc{Finalize}\}$ for $t \geq 0$. Procedures P_1 to P_t and Finalize might require some input parameters. We implicitly assume that boolean flags are initialized to *false*, numerical types are initialized to 0, sets are initialized to $\emptyset$, while strings are initialized to the empty string ϵ. An adversary $\mathcal{A}$ is *run in game* G if $\mathcal{A}$ calls Initialize. During the game $\mathcal{A}$ may run some procedure P_i as often as allowed by the game.

For each game in this paper, the "Open" procedure may be called an arbitrary number of times, while every other procedure is called once during the execution.

The interface of the game is provided by the *challenger*. If $\mathcal{A}$ calls P, the output of P is returned to $\mathcal{A}$, except for the Finalize procedure. On $\mathcal{A}$'s call of Finalize the game ends and outputs whatever Finalize returns. Let $\mathsf{G}^{\mathcal{A}} \Rightarrow \mathsf{out}$ denote the event that G runs $\mathcal{A}$ and outputs out. The *advantage* $\mathbf{Adv}(\mathsf{G}^{\mathcal{A}}, \mathsf{H}^{\mathcal{A}})$ of $\mathcal{A}$ in distinguishing games G and H is defined as $\big|\Pr[\mathsf{G}^{\mathcal{A}} \Rightarrow 1] - \Pr[\mathsf{H}^{\mathcal{A}} \Rightarrow 1]\big|$. We let $\mathtt{Bad}$ denote the event that a boolean flag $\mathtt{Bad}$ was set to *true* during the execution of some game.

2.2 Public-Key Encryption Schemes

A public-key encryption scheme consists of three PPT algorithms. Gen generates a key pair $(pk, sk) \leftarrow \mathsf{Gen}(1^\lambda)$ on input 1^λ. The public key pk implicitly contains 1^λ and defines three finite sets: the message space $\mathcal{M}$, the randomness space $\mathcal{R}$, and the ciphertext space $\mathcal{C}$. Given pk, a message $m \in \mathcal{M}$ and randomness $r \in \mathcal{R}$, Enc outputs an encryption $c = \mathsf{Enc}_{pk}(m; r) \in \mathcal{C}$ of m under pk. The decryption algorithm Dec takes a secret key sk and a ciphertext $c \in \mathcal{C}$ as input and outputs a message $m = \mathsf{Dec}_{sk}(c) \in \mathcal{M}$, or a special symbol $\perp \notin \mathcal{M}$ indicating that c is not a valid ciphertext. In the following we let $\mathsf{PKE} = (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ denote a public-key encryption scheme.

We require PKE to be correct: for all security parameters λ, for all $(pk, sk) \leftarrow \mathsf{Gen}(1^\lambda)$, and for all $m \in \mathcal{M}$ we have $\Pr[\mathsf{Dec}_{sk}(\mathsf{Enc}_{pk}(m; r)) = m] = 1$ where the probability is taken over the choice of r. We apply Enc and Dec to message vectors $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$ and randomness $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n)$ as $\mathsf{Enc}(\mathbf{m}; \mathbf{r}) := (\mathsf{Enc}(\mathbf{m}_1; \mathbf{r}_1), \ldots, \mathsf{Enc}(\mathbf{m}_n; \mathbf{r}_n))$.

Procedure INITIALIZE	**Procedure** CHALLENGE($\mathbf{m}^0, \mathbf{m}^1$)	**Procedure** FINALIZE(b')
$(pk, sk) \leftarrow \mathsf{Gen}(1^\lambda)$ Return pk	$\mathbf{c} \leftarrow \mathsf{Enc}_{pk}(\mathbf{m}^b)$ Return $\mathbf{c}$	Return b'

Fig. 1. Game $\mathsf{mult\text{-}IND\text{-}CPA}_{\mathsf{PKE},b}$; $\mathcal{B}_{\mathsf{mult}}$ must submit $\mathbf{m}^0, \mathbf{m}^1 \in \mathcal{M}^s$

2.3 IND-CPA and mult-IND-CPA Security

We revise the standard notion of IND-CPA security and give a definition of indistinguishable ciphertext vectors under chosen-plaintext attacks that will allow for cleaner proofs of our results.

Definition 1 (mult-IND-CPA security). *For* PKE*, an adversary* $\mathcal{B}_{\mathsf{mult}}$*,* $s \in \mathbb{N}$ *and a bit* b *we consider game* $\mathsf{mult\text{-}IND\text{-}CPA}^{\mathcal{B}_{\mathsf{mult}}}_{\mathsf{PKE},b}$ *as given in Fig. 1.* $\mathcal{B}_{\mathsf{mult}}$ *may only submit message vectors* $\mathbf{m}^0$*,* $\mathbf{m}^1 \in \mathcal{M}^s$*. To* PKE*,* $\mathcal{B}_{\mathsf{mult}}$ *and* λ *we associate the following advantage function*

$$\mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) := \mathbf{Adv}\big(\mathsf{mult\text{-}IND\text{-}CPA}^{\mathcal{B}_{\mathsf{mult}}}_{\mathsf{PKE},0}, \mathsf{mult\text{-}IND\text{-}CPA}^{\mathcal{B}_{\mathsf{mult}}}_{\mathsf{PKE},1}\big) \ .$$

PKE *is* mult-IND-CPA *secure if* $\mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda)$ *is negligible for all* PPT *adversaries* $\mathcal{B}_{\mathsf{mult}}$*.*

For an adversary $\mathcal{B}_{\mathsf{CPA}}$, we obtain the definition of IND-CPA security by letting $s := 1$ and write $\mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda)$ instead of $\mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda)$. A standard hybrid argument proves the following lemma.

Lemma 2. *For any adversary* $\mathcal{B}_{\mathsf{mult}}$ *sending message vectors from* $\mathcal{M}^s$ *to the* mult-IND-CPA *game there exists an* IND-CPA *adversary* $\mathcal{B}_{\mathsf{CPA}}$ *with roughly the same running time as* $\mathcal{B}_{\mathsf{mult}}$ *such that*

$$\mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) \leq s \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \ .$$

2.4 IND-SO-CPA Security

In this section we recall an indistinguishability-based definition for selective opening security under chosen-plaintext attacks and discuss the existing notions of SO security.

Definition 3 (Efficiently resamplable distribution). *Let* $\mathcal{M}$ *be a finite set. A family of distributions* $\{\mathfrak{D}_\lambda\}_{\lambda\in\mathbb{N}}$ *over* $\mathcal{M}^n = \mathcal{M}^{n(\lambda)}$ *is* efficiently resamplable *if the following properties hold for every* $\lambda \in \mathbb{N}$*:*

Length consistency. *For every* $i \in [n]$*:* $\Pr_{\mathbf{m}^1,\mathbf{m}^2\leftarrow\mathfrak{D}_\lambda}\left[|\mathbf{m}^1_i| = |\mathbf{m}^2_i|\right] = 1$.

Resamplability. *There exists a* PPT *resampling algorithm* $\mathsf{Resamp}_{\mathfrak{D}_\lambda}(\cdot,\cdot)$ *that runs on* $(\mathbf{m}, \mathcal{I})$ *for* $\mathbf{m} \in \mathcal{M}^n$*,* $\mathcal{I} \subseteq [n]$ *and outputs a* $\mathfrak{D}_\lambda$*-distributed vector* $\mathbf{m}' \in \mathcal{M}^n$ *conditioned on* $\mathbf{m}'_{\mathcal{I}} = \mathbf{m}_{\mathcal{I}}$*.*

Procedure INITIALIZE	**Procedure** OPEN(i)
$(pk, sk) \leftarrow \mathsf{Gen}(1^\lambda)$	$\mathcal{I} := \mathcal{I} \cup \{i\}$
Return pk	Return $(\mathbf{m}_i^0, \mathbf{r}_i)$
Procedure ENC($\mathfrak{D}, \mathsf{Resamp}_{\mathfrak{D}}$)	**Procedure** CHALLENGE
$\mathbf{m}^0 \leftarrow \mathfrak{D}$	$\mathbf{m}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, \mathcal{I})$
$\mathbf{r} \leftarrow\!\!\$\ \mathcal{R}^n$	Return $\mathbf{m}^b$
$\mathbf{c} = \mathsf{Enc}_{pk}(\mathbf{m}^0; \mathbf{r})$	**Procedure** FINALIZE(b')
Return $\mathbf{c}$	Return b'

Fig. 2. Game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},b}$

A class of families of distributions $\mathcal{D}$ is efficiently resamplable if every family $\{\mathfrak{D}_\lambda\}_{\lambda\in\mathbb{N}} \in \mathcal{D}$ is efficiently resamplable.

Since the security parameter uniquely specifies an element of a family $\mathfrak{D}_\lambda$ we write $\mathfrak{D}$ instead of $\mathfrak{D}_\lambda$ whenever the security parameter is already fixed.

Definition 4. *For* PKE*, a bit b, an adversary $\mathcal{A}_{\mathsf{SO}}$ and a class of families of distributions $\mathcal{D}$ over $\mathcal{M}^n$ we consider game $\mathsf{IND\text{-}SO\text{-}CPA}^{\mathcal{A}_{\mathsf{SO}}}_{\mathsf{PKE},b}$ in Fig. 2. Run in the game, $\mathcal{A}_{\mathsf{SO}}$ calls* ENC *once right after* INITIALIZE *and has to submit $\mathfrak{D} \in \mathcal{D}$ along with a* PPT *resampling algorithm $\mathsf{Resamp}_{\mathfrak{D}}$. $\mathcal{A}_{\mathsf{SO}}$ may call* OPEN *multiple times and invokes* CHALLENGE *once after its last* OPEN *query before calling* FINALIZE*. We define the advantage of $\mathcal{A}_{\mathsf{SO}}$ run in game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},b}$ as*

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) := \mathbf{Adv}\big(\mathsf{IND\text{-}SO\text{-}CPA}^{\mathcal{A}_{\mathsf{SO}}}_{\mathsf{PKE},0}, \mathsf{IND\text{-}SO\text{-}CPA}^{\mathcal{A}_{\mathsf{SO}}}_{\mathsf{PKE},1}\big)\ .$$

PKE is IND-SO-CPA secure w.r.t. $\mathcal{D}$ *if* $\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda)$ *is negligible for all* PPT $\mathcal{A}_{\mathsf{SO}}$.

NOTIONS OF SELECTIVE OPENING SECURITY. Def. 4 is in the spirit of [2] but we allow for adaptive corruptions and let the adversary choose the distribution, as done by Böhl et al. [4]. The latter renamed IND-SO-CPA to *weak* IND-SO-CPA and introduced a strictly stronger notion, called *full* IND-SO-CPA, where $\mathcal{A}_{\mathsf{SO}}$ may submit any distribution (even one not efficiently resamplable) and need not provide a resampling algorithm.[1] We consider the name *weak* IND-SO-CPA unfortunate and simply refer to the security notion in Def. 4 as IND-SO-CPA security.

3 Selective Opening for Graph-Induced Distributions

This section considers graph-induced distributions and identifies connectivity properties so that IND-CPA entails IND-SO-CPA security. We introduce some

[1] E.g., for a one-way function OWF a distribution $(m,\mathsf{OWF}(m))$ may not support efficient resampling.

notation in Section 3.1. Sections 3.2 and 3.3 discuss a hybrid argument that considers the connected components of $G_{\overline{\mathcal{I}}}$, switching one of them from *sampled* to *resampled* in each transition. Section 3.4 discusses a different hybrid argument that will allow for tighter proofs if the distribution-inducing graph is acyclic.

3.1 Graphs

A *directed graph* G consists of a set of vertices V, identified with $[n]$ for $n > 0$ and a set of edges $E \subseteq V^2 \setminus \{(v,v)\colon v \in V\}$, i.e. we do not allow loops. G is *undirected* if $(v_2, v_1) \in E$ for each $(v_1, v_2) \in E$. For $V' \subseteq V$ let $G_{V'} := (V', E')$ denote the *induced subgraph* of G where $E' := E \cap (V')^2$. For $G = (V, E)$ we obtain its *undirected version*, $G^{\leftrightarrow} = (V, E^{\leftrightarrow})$ where $E^{\leftrightarrow} \supseteq E$ is obtained by adding the minimum number of edges to E so that the graph becomes undirected. For $V' \subseteq V$ let $N(V') := \{v \in V \setminus V'\colon \exists v' \in V' \text{ s.t. } (v,v') \in E^{\leftrightarrow}\}$ denote the *(open) neighborhood* of V' in G. For a vertex v, we denote by $P(v) = \{j\colon (j,v) \in E\}$ the set of its *parents*.

A *path* from v_1 to v_ℓ in G is a list of at least two vertices $(v_1, \ldots, v_\ell)$ where $v_i \in V$ for $i \in [\ell]$ and $(v_i, v_{i+1}) \in E$ for all $i \in [\ell - 1]$. If there is a path from u to v then u is a *predecessor* of v. Let $\mathsf{pred}(v)$ denote the set of all predecessors of v. A *cycle* is a path where $v_\ell = v_1$. If G contains no cycles, it is *acyclic*. A directed, acyclic graph is called DAG.

A non-empty subset $V' \subseteq V$ is *connected* in G if for every pair of distinct vertices $(v_1, v_2) \in V'$ there exists a path form v_1 to v_2 in $G^{\leftrightarrow}$. G *is connected* if V is connected in G. G is *disconnected* if G is not connected. We assume G to be connected if not stated otherwise. A (set-)maximal connected set of vertices of G is called *connected component*.

Notational convention. We do not distinguish between the i-th message of an n-message vector and vertex i in a graph on n vertices.

We start with defining Markov distributions, which are distributions on vectors of random variables reflecting processes, that is, variables with higher indices depend on ones with lower indices. A distribution is Markov if it is *memoryless* in the sense that all relevant information for the distribution of a value $\mathbf{M}_i$ is already present in $\mathbf{M}_{i-1}$, although the latter itself depends on its predecessor.

Definition 5. *Let $\{\mathfrak{D}_\lambda\}_{\lambda\in\mathbb{N}}$ be a family of distributions over $\mathcal{M}^n$. Let $\mathbf{M} = (\mathbf{M}_1, \ldots, \mathbf{M}_n)$ denote a vector of $\mathcal{M}$-valued random variables. We say $\{\mathfrak{D}_\lambda\}_{\lambda\in\mathbb{N}}$ is* Markov *if the following holds for all $\lambda \in \mathbb{N}$ and all $\mathbf{m} \in \mathcal{M}^n$:*

$$\Pr_{\mathbf{M}\leftarrow\mathfrak{D}_\lambda}\left[\mathbf{M}_i = \mathbf{m}_i \,\middle|\, \bigwedge_{j=1}^{i-1} \mathbf{M}_j = \mathbf{m}_j\right] = \Pr_{\mathbf{M}\leftarrow\mathfrak{D}_\lambda}\left[\mathbf{M}_i = \mathbf{m}_i \,\middle|\, \mathbf{M}_{i-1} = \mathbf{m}_{i-1}\right] \ .$$

Markov distribution can be seen as "induced" by a chain graph $\mathbf{M}_1 \to \mathbf{M}_2 \to \ldots \to \mathbf{M}_n$, where edges represent dependencies. We will now generalize this to arbitrary graphs and still require (a generalization of) "memorylessness".

We say that a graph G induces a distribution $\mathfrak{D}$ if whenever the distribution of $\mathbf{M}_j$ depends on $\mathbf{M}_i$ then there is a path from i to j in G. As for Markov distributions, we require that the distribution of a message only depends on its parents; in particular, for all $\lambda \in \mathbb{N}$, all $j \in [n]$ and $\mathbf{M} = (\mathbf{M}_1, \ldots, \mathbf{M}_n) \leftarrow \mathfrak{D}_\lambda$ the distribution of $\mathbf{M}_j$ only depends on its parents in G_λ, i.e. the set $P(j)$, rather than all its predecessors $\mathsf{pred}(j)$.

Definition 6 (Graph-induced distribution). *Let $\{\mathfrak{D}_\lambda\}_{\lambda\in\mathbb{N}}$ be a family of distributions over $\mathcal{M}^n$ and let $\{G_\lambda\}_{\lambda\in\mathbb{N}}$ be a family of graphs on n vertices. We say that $\{\mathfrak{D}_\lambda\}_{\lambda\in\mathbb{N}}$ is $\{G_\lambda\}_{\lambda\in\mathbb{N}}$-*induced *if the following holds for all $\lambda \in \mathbb{N}$:*

- *For all $i \neq j \in [n]$ if for $\mathfrak{D}_\lambda$ the distribution of $\mathbf{M}_j$ depends on $\mathbf{M}_i$ then there is a path from i to j in G_λ.*
- *For all $j \in [n]$ and all $\mathbf{m} \in \mathcal{M}^n$ we have*

$$\Pr_{\mathbf{M}\leftarrow\mathfrak{D}_\lambda}\Big[\mathbf{M}_j = \mathbf{m}_j \,\Big|\, \bigwedge_{i\in\mathsf{pred}(j)} \mathbf{M}_i = \mathbf{m}_i\Big] = \Pr_{\mathbf{M}\leftarrow\mathfrak{D}_\lambda}\Big[\mathbf{M}_j = \mathbf{m}_j \,\Big|\, \bigwedge_{i\in P(j)} \mathbf{M}_i = \mathbf{m}_i\Big] \ .$$

We demand that for any $\lambda \in \mathbb{N}$ one can efficiently reconstruct G_λ from $\mathfrak{D}_\lambda$.

As with a family of distributions, we drop the security parameter and say that $\mathfrak{D}$ *is G-induced* whenever λ is already fixed. Note that G may contain cycles and may be undirected. Further note that Markov distributions can be seen as graph-induced distributions where the graph $G = (V, E)$ is a chain on n vertices, that is, $V = [n]$ and $E = \{(i-1, i) \colon i \in [n]\}$.

Although our proof ideas can be applied to disconnected graphs directly, Sections 3.2 – 3.4 consider *connected graphs* for simplicity. A hybrid argument over the connected components of a graph as given in Section 3.5 extends all our results to disconnected graphs.

3.2 A Bound Using Connected Subgraphs

Definition 7 (Number of connected subgraphs). *Let $G = (V, E)$. We define the* number of connected subgraphs *of G:*

$$S(G) := |\{V' \subseteq V \colon V' \text{ connected}\}| \ .$$

For example, for a chain graph on n vertices we have $S(G) = \frac{1}{2} \cdot n \cdot (n+1)$ and for the complete graph C_n on n vertices we have $S(C_n) = 2^n - 1$.

Theorem 8. *Let* PKE *be* IND-CPA *secure. Then* PKE *is* IND-SO-CPA *secure w.r.t. the class of efficiently resamplable and G-induced distribution families over $\mathcal{M}^n$ where $S(G) = \mathsf{poly}(n)$ and G is connected.*

Precisely, for any adversary $\mathcal{A}_{\mathsf{SO}}$ run in game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE}}$ *there exists an* $\mathsf{IND\text{-}CPA}_{\mathsf{PKE}}$ *adversary $\mathcal{B}_{\mathsf{CPA}}$ with roughly the running time of $\mathcal{A}_{\mathsf{SO}}$ plus two executions of* Resamp *such that*

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \leq n \cdot (n-1) \cdot S(G_\lambda) \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \ .$$

Procedure CHALLENGE
$\mathbf{m}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, \mathcal{I})$
$\mathbf{m}_i := \begin{cases} \mathbf{m}_i^1 & \text{for } i \in \bigcup_{j=1}^{k} C_j \\ \mathbf{m}_i^0 & \text{else} \end{cases}$
Return $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$

Fig. 3. CHALLENGE procedure of hybrid game H_k. C_i denotes the i-th connected component of $G_{\overline{\mathcal{I}}}$. The challenge vector contains resampled messages in the first k batches $C_1, \ldots, C_k$ while the other messages remain sampled.

PROOF IDEA. Recall game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},b}$ given in Fig. 2. During CHALLENGE the game sends $\mathbf{m}^b$, where $\mathbf{m}^0_{\overline{\mathcal{I}}}$ consists of messages sampled at the beginning, while $\mathbf{m}^1_{\overline{\mathcal{I}}}$ is resampled (conditioned on $\mathbf{m}^1_{\mathcal{I}} = \mathbf{m}^0_{\mathcal{I}}$). We will define hybrid games $\mathsf{H}_0, \mathsf{H}_1, \ldots, \mathsf{H}_n$. For this, let $\mathcal{S} \subseteq 2^V$ denote all the connected subgraphs of G. We have $|\mathcal{S}| = S(G)$.

Note that $G_{\overline{\mathcal{I}}}$ consists of connected components $C_1, \ldots, C_{n'} \subseteq \mathcal{S}$ for some $n' \leq n-1$. (This upper bound is attained by the star graph when I consists of the internal node.) We assume those components to be ordered, e.g. by the smallest vertex contained in each.

Thus, if $b = 1$ in game $\mathsf{IND\text{-}SO\text{-}CPA}$ then the challenger can resample $\mathbf{m}^1_{\overline{\mathcal{I}}}$ in n' batches $\mathbf{m}^1_{C_1}, \ldots, \mathbf{m}^1_{C_{n'}}$ (as $\overline{\mathcal{I}} = \bigcup_{i=1}^{n'} C_i$). Moreover, each batch $\mathbf{m}^1_{C_i}$ can be resampled *independently*, i.e., as a function of $\mathbf{m}^0_{\mathcal{I}}$ and $\mathfrak{D}$, but not $\mathbf{m}^1_{C_j}$, $j \neq i$.

Proof (Theorem 8). For $k = 0, \ldots, n$ we define hybrid game H_k as a modified game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE}}$, in which the messages of the first k batches $C_1, \ldots, C_k$ are resampled during CHALLENGE while the remaining batches stay sampled.

Every procedure except CHALLENGE remains as in Def. 4, and CHALLENGE is given in Fig. 3. Clearly, H_0 is the (real) game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},0}$ and $\mathsf{H}_{n'}$ for some $n' \leq n-1$ is the (random) game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},1}$. Note that for $k, j \in [n', n]$ hybrids H_k and H_j are identical. We have

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) = \mathbf{Adv}\big(\mathsf{H}_0^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{n'}^{\mathcal{A}_{\mathsf{SO}}}\big) \leq \sum_{k=0}^{n'-1} \mathbf{Adv}\big(\mathsf{H}_k^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{k+1}^{\mathcal{A}_{\mathsf{SO}}}\big) \ .$$

We now upper-bound the distance between two consecutive hybrids using the following lemma.

Lemma 9. *For every adversary $\mathcal{A}_{\mathsf{SO}}$ that distinguishes hybrids H_k and H_{k+1} there exists a* mult-IND-CPA *adversary $\mathcal{B}_{\mathsf{mult}}$ with roughly the running time of $\mathcal{A}_{\mathsf{SO}}$ plus two executions of* Resamp *such that*

$$\mathbf{Adv}\big(\mathsf{H}_k^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{k+1}^{\mathcal{A}_{\mathsf{SO}}}\big) \leq S(G) \cdot \mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) \ .$$

Proof. We construct adversary $\mathcal{B}_{\mathsf{mult}}$ as follows (cf. Fig. 4):

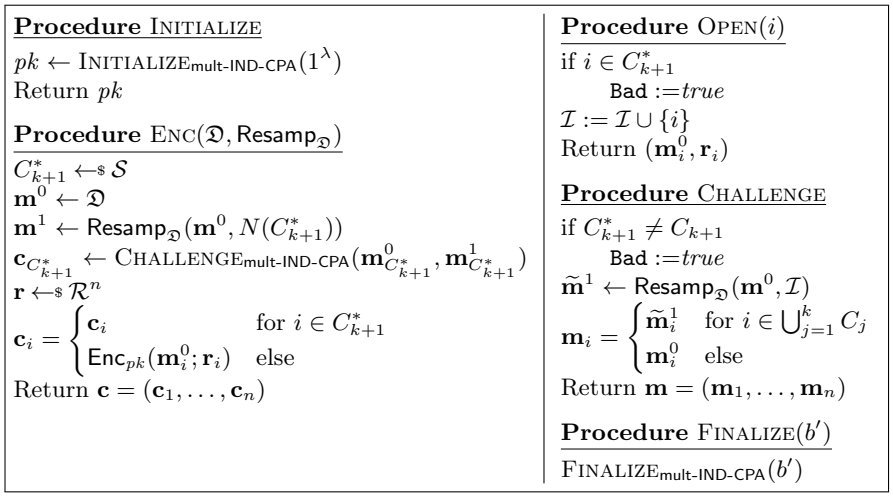

Fig. 4. $\mathcal{A}_{\mathsf{SO}}$'s game interface as provided by $\mathcal{B}_{\mathsf{mult}}$ run in game mult-IND-CPA. $\mathcal{B}_{\mathsf{mult}}$ interpolates between hybrids H_k, H_{k+1} for $k \in [0, n-1]$.

$\mathcal{B}_{\mathsf{mult}}$ forwards pk to $\mathcal{A}_{\mathsf{SO}}$ and picks $C^*_{k+1} \leftarrow\!\!\$\ \mathcal{S}$ uniformly at random (trying to guess C_{k+1}) after receiving $(\mathfrak{D}, \mathsf{Resamp}_{\mathfrak{D}})$. $\mathcal{B}_{\mathsf{mult}}$ samples $\mathbf{m}^0 \leftarrow \mathfrak{D}$ and resamples $\mathbf{m}^1$ keeping the neighborhood of C^*_{k+1} fixed. It submits $(\mathbf{m}^0_{C^*_{k+1}}, \mathbf{m}^1_{C^*_{k+1}})$ to its mult-IND-CPA challenger, obtains ciphertexts for positions in C^*_{k+1}, picks randomness and uses it to encrypt each message in $\overline{C^*_{k+1}}$. $\mathcal{B}_{\mathsf{mult}}$ sends $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ to $\mathcal{A}_{\mathsf{SO}}$, embedding its challenge at positions C^*_{k+1} and answers opening queries honestly if they do not occur on C^*_{k+1}. If $\mathcal{A}_{\mathsf{SO}}$ issues such a query, $\mathcal{B}_{\mathsf{mult}}$ cannot answer and sets Bad $:= true$ since it guessed C_{k+1} wrong. During Challenge, $\mathcal{B}_{\mathsf{mult}}$ verifies that it guessed C_{k+1} correctly and sets Bad $:= true$ if not. $\mathcal{B}_{\mathsf{mult}}$ resamples messages $\widetilde{\mathbf{m}}^1$ that are sent in the first k batches while messages from $\mathbf{m}^0$ are sent in every other position. $\mathcal{B}_{\mathsf{mult}}$ outputs $\mathcal{A}_{\mathsf{SO}}$'s output.

In the following we use $\mathbf{m} \equiv \mathbf{m}'$ if $\mathbf{m}$ and $\mathbf{m}'$, interpreted as random variables, are identically distributed where the probability is taken over all choices in the computation of $\mathbf{m}$, $\mathbf{m}'$, respectively.

Assume, $\mathcal{B}_{\mathsf{mult}}$ guessed correctly, i.e. $C^*_{k+1} = C_{k+1}$. Clearly, $\mathcal{B}_{\mathsf{mult}}$ perfectly simulates hybrids H_k and H_{k+1} for messages and ciphertexts at positions in $\overline{C_{k+1}}$. Run in $\mathsf{mult\text{-}IND\text{-}CPA}_{\mathsf{PKE},0}$, $\mathcal{B}_{\mathsf{mult}}$ obtains $\mathsf{Enc}_{pk}(\mathbf{m}^0_{C_{k+1}})$ and $\mathcal{A}_{\mathsf{SO}}$ therefore receives encryptions of sampled messages. During Challenge the $(k+1)$-th batch contains sampled messages $\mathbf{m}^0_{C_{k+1}}$, thus $\mathcal{B}_{\mathsf{mult}}$ perfectly simulates hybrid H_k.

When $\mathcal{B}_{\mathsf{mult}}$ is run in $\mathsf{mult\text{-}IND\text{-}CPA}_{\mathsf{PKE},1}$, $\mathcal{A}_{\mathsf{SO}}$ obtains encryptions of *resampled* messages $\mathsf{Enc}_{pk}(\mathbf{m}^1_{C_{k+1}})$ while it expects encrypted *sampled* messages: $\mathsf{Enc}_{pk}(\mathbf{m}^0_{C_{k+1}})$. During Challenge $\mathcal{A}_{\mathsf{SO}}$ expects *resampled* messages $\widetilde{\mathbf{m}}^1_{C_{k+1}}$ but obtains *sampled* $\mathbf{m}^0_{C_{k+1}}$. Thus, the *sampled* and *resampled* messages change roles on C_{k+1}.

However, they are equally distributed, i.e., $\mathbf{m}^0_{C_{k+1}} \equiv \mathbf{m}^1_{C_{k+1}}$ since the messages in $N(C_{k+1})$ were fixed when resampling $\mathbf{m}^1$ and the distribution of messages in C_{k+1} depends on $\mathfrak{D}$ and messages in positions $N(C_{k+1})$ only. Likewise, $\mathbf{m}^1_{C_{k+1}} \equiv \widetilde{\mathbf{m}}^1_{C_{k+1}}$ for $\mathbf{m}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, N(C_{k+1}))$ and $\widetilde{\mathbf{m}}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, \mathcal{I})$ since the distribution of messages in C_{k+1} solely depends on $\mathfrak{D}$ and messages in $N(C_{k+1}) \subseteq \mathcal{I}$ and $\mathcal{A}_{\mathsf{SO}}$'s view is identical to hybrid H_{k+1}. We have

$$\Pr[\mathsf{mult\text{-}IND\text{-}CPA}^{\mathcal{B}_{\mathsf{mult}}}_{\mathsf{PKE},0} \Rightarrow 1] = \Pr[\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_k \Rightarrow 1 \wedge \overline{\mathtt{Bad}}] \text{ and}$$

$$\Pr[\mathsf{mult\text{-}IND\text{-}CPA}^{\mathcal{B}_{\mathsf{mult}}}_{\mathsf{PKE},1} \Rightarrow 1] = \Pr[\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k+1} \Rightarrow 1 \wedge \overline{\mathtt{Bad}}] \ .$$

Observe that $\mathtt{Bad}$ does not happen when $\mathcal{B}_{\mathsf{mult}}$ guessed C_{k+1} correctly. Since $\overline{\mathtt{Bad}}$ is independent of $\mathcal{A}_{\mathsf{SO}}$'s output in a hybrid and $|\mathcal{S}| = S(G)$, we have

$$\mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) \geq \frac{1}{S(G)} \cdot \mathbf{Adv}\big(\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_k, \mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k+1}\big) \ ,$$

which concludes the proof. □

We proceed with the proof of Theorem 8. Using Lemma 9 we have

$$\begin{aligned}\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) &\leq \sum_{k=0}^{n'-1} \mathbf{Adv}(\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_k, \mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k+1}) \\ &\leq \sum_{k=0}^{n'-1} S(G_\lambda) \cdot \mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) \ .\end{aligned}$$

$\mathcal{B}_{\mathsf{mult}}$ sends message vectors of length $|C^*_{k+1}| \leq n$ to its $\mathsf{mult\text{-}IND\text{-}CPA}$ challenger. Using Lemma 2, we have

$$\leq \sum_{k=0}^{n'-1} n \cdot S(G_\lambda) \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \leq n \cdot (n-1) \cdot S(G_\lambda) \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \ ,$$

since $n' \leq n-1$, which completes the proof of Theorem 8. □

Markov Distributions. Markov distributions (Def. 5) are induced by the chain graph $(V = [n], E = \{(i-1, i) : i \in [n]\})$, for which $S(G) = \frac{1}{2} \cdot n \cdot (n+1)$. We thus immediately obtain the following corollary from Theorem 8.

Corollary 10. *Let* PKE *be* $\mathsf{IND\text{-}CPA}$ *secure. Then* PKE *is* $\mathsf{IND\text{-}SO\text{-}CPA}$ *secure w.r.t. efficiently resamplable Markov distributions over* $\mathcal{M}^n$*.*

Precisely, for any adversary $\mathcal{A}_{\mathsf{SO}}$ *run in game* $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE}}$ *there exists an* $\mathsf{IND\text{-}CPA}_{\mathsf{PKE}}$ *adversary* $\mathcal{B}_{\mathsf{CPA}}$ *with roughly the running time of* $\mathcal{A}_{\mathsf{SO}}$ *plus two executions of* Resamp *such that*

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \leq \tfrac{1}{2} \cdot n^2 \cdot (n^2 - 1) \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \ .$$

3.3 A Bound Using the Maximum Border

Definition 11 (Maximum border). *Let $G = (V, E)$. We define the* maximum border *of G as the maximal size of the neighborhood of any connected subgraph in G.*

$$B(G) := \max\left\{\, |N(V')| \;:\; V' \subseteq V \text{ connected} \right\} \ .$$

For example, if G is an n-path for $n \geq 3$ then $B(G) = 2$. For the complete graph or star graph on n vertices we have $B(G) = n - 1$. Notice that $B(G) < n$.

In the reduction in Section 3.2 we guessed a connected component in $G_{\overline{\mathcal{I}}}$ that would be switched from sampled to resampled in a hybrid transition. Alternatively, we can guess a connected component in $G_{\overline{\mathcal{I}}}$ via its neighborhood. The following theorem expresses $S(G)$ in terms of $B(G)$.

Theorem 12. *Let G be a connected graph. Then the following bound on $S(G)$ holds:*

$$S(G) \leq \frac{2}{(B(G)-1)!} \cdot n^{B(G)} \quad \text{for all} \quad 0 < B(G) \leq \frac{n-2}{3} \ .$$

We begin with a simple observation before proving the theorem.

Lemma 13. *Let $G = (V, E)$ and $V_1 \neq V_2$ each of them connected in G such that $N(V_1) = N(V_2)$. Then $V_1 \cap V_2 = \emptyset$.*

Proof. Assume $V_1 \cap V_2 \neq \emptyset$. As $V_1 \neq V_2$ we have $V_1 \setminus V_2 \neq \emptyset$ without loss of generality. Because V_1 is connected, there exist vertices $v_\cap \in V_1 \cap V_2$ and $v_1 \in V_1 \setminus V_2$ such that $(v_1, v_\cap) \in E^{\leftrightarrow}$. Since $v_1 \notin V_2$, $v_\cap \in V_2$ and $(v_1, v_\cap) \in E^{\leftrightarrow}$, we see that $v_1 \in N(V_2)$. As $N(V_2) = N(V_1)$ it follows that $v_1 \in N(V_1)$; a contradiction since $v_1 \in V_1$. □

Proof (Theorem 12). Let $B := B(G)$. We have

$$S(G) = \sum_{i=0}^{B} \left|\left\{V' \subseteq V \colon V' \text{ connected} \wedge |N(V')| = i\right\}\right| \ .$$

For $i = 0$ we count the connected components of G.

$$= 1 + \sum_{i=1}^{B} \left|\left\{V' \subseteq V \colon V' \text{ connected} \wedge |N(V')| = i\right\}\right|$$

$$= 1 + \sum_{i=1}^{B} \sum_{\substack{V_i \subseteq V \\ |V_i| = i}} \left|\left\{V' \subseteq V \colon V' \text{ connected} \wedge N(V') = V_i\right\}\right| \ .$$

Let $V_i \subseteq V$ be non-empty and $\{V' \subseteq V \colon V' \text{ connected} \wedge N(V') = V_i\} = \{V'_1, \ldots, V'_k\}$ for appropriate k. Applying Lemma 13 to $V'_1, \ldots, V'_k$, we see that

those sets V'_j are pairwise disjoint. Fix any vertex $v_i \in V_i$. Since $N(V'_j) = V_i$ for $j \in [k]$ and all V'_j are pairwise disjoint, there exists at least one vertex v'_j in each V'_j such that $(v'_j, v_i) \in E$ for all $j \in [k]$. Thus, $N(v_i) \geq k$, i.e. $B \geq k$. Hence, $k \leq B$ for given B and we obtain an upper bound for the number of possible sets V' for each fixed V_i. It follows

$$S(G) \leq 1 + \sum_{i=1}^{B} \sum_{\substack{V_i \subseteq V \\ |V_i|=i}} B = 1 + B \cdot \sum_{i=1}^{B} \binom{n}{i} \leq B \cdot \sum_{i=0}^{B} \binom{n}{i} \ . \tag{1}$$

To bound the sum in (1) we use the geometric series and upper-bound the quotient of two consecutive binomial coefficients by $\frac{1}{2}$:

$$\frac{\binom{n}{i}}{\binom{n}{i+1}} = \frac{i+1}{n-i} \leq \frac{1}{2} \Leftrightarrow i \leq \frac{n-2}{3} \ .$$

Hence

$$B \cdot \sum_{i=0}^{B} \binom{n}{i} \leq B \cdot \sum_{i=0}^{B} \frac{1}{2^i} \binom{n}{B} \leq B \cdot \binom{n}{B} \cdot \sum_{i=0}^{\infty} \frac{1}{2^i} \leq 2 \cdot B \cdot \frac{n^B}{B!} = \frac{2}{(B-1)!} \cdot n^B$$

for $B(G) \leq \frac{n-2}{3}$, which concludes the proof. □

Theorems 8 and 12 together now yield the following corollary.

Corollary 14. *Let* PKE *be* IND-CPA *secure. Then* PKE *is* IND-SO-CPA *secure w.r.t. the class of efficiently resamplable and G-induced distribution families over $\mathcal{M}^n$ where $B(G) = \mathsf{const}$, $n \geq 3 \cdot B(G) + 2$ and G is connected.*

Concretely, for any adversary $\mathcal{A}_{\mathsf{SO}}$ in game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE}}$ *there exists an* $\mathsf{IND\text{-}CPA}_{\mathsf{PKE}}$ *adversary $\mathcal{B}_{\mathsf{CPA}}$ with roughly the running time of $\mathcal{A}_{\mathsf{SO}}$ plus two executions of* Resamp *such that*

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \leq \frac{2 \cdot (n-1)}{(B(G_\lambda)-1)!} \cdot n^{B(G_\lambda)+1} \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \ .$$

Since Corollary 14 ensures a polynomial loss in the reduction for $B(G) = \mathsf{const}$ and we are interested in asymptotic statements, we do not consider the restriction to $n \geq 3 \cdot B(G) + 2$ grave. One can easily obtain a version of Theorem 12 that is weaker by a factor of roughly $B(G)$ but holds for all $B(G) < n$. To this end one bounds the sum of binomial coefficients in (1) in terms of the incomplete upper gamma function Γ to get

$$\sum_{i=1}^{B} \binom{n}{i} \leq \sum_{i=1}^{B} \frac{n^i}{i!} = \frac{e^n \Gamma(B+1, n)}{B!} - 1 \ .$$

Using a nice bound on Γ due to [11] that can be found in [5] we obtain a bound for $B(G) < n$.

Procedure CHALLENGE
$\mathbf{m}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, [k+1, n] \cup \mathcal{I})$
Return $\mathbf{m}^b$

Fig. 5. CHALLENGE procedure of hybrid game H_k. For $k = n$ we have $[n+1, n] = \emptyset$.

Think of a direct reduction for proving Corollary 14 as implicitly guessing C_{k+1} via guessing $N(C_{k+1})$ by picking up to $B(G)$ vertices in G and guessing one of at most $B(G)$ connected subgraphs that have the guessed neighborhood.

Note that Corollary 14 cannot provide a tighter bound on the loss than Theorem 8. In particular, there are (even connected) graphs for which Theorem 8 ensures an at most polynomial loss, while Corollary 14 does not. For instance, let G be the star graph on $\log n$ vertices attached to the chain graph of $n - \log n$ vertices, then $S(G) = \mathsf{poly}(n)$, but $B(G) > \mathsf{const}$.

3.4 A Tighter Reduction for Acyclic Graphs

While we considered graph-induced distributions for arbitrary graphs in Sections 3.2 and 3.3, we now consider DAG-induced distributions for which we obtain a tighter reduction than what is guaranteed by Corollary 14.

For a DAG G we require that the vertices are semi-ordered in such a way that there is no directed path from i to j for $i < j$. Such an ordering always exists as G has no cycles. Note that the dependencies now go the other way as for Markov distributions, but this will allow us to replaced sampled messages by resampled ones from left to right as in the previous hybrids. We will traverse dependencies *backwards*, that is, if message $\mathbf{m}_i$ depends on $\mathbf{m}_j$ then $\mathbf{m}_i$ is switched from *sampled* to *resampled* before $\mathbf{m}_j$ is switched. So, as in the previous proofs, messages $\mathbf{m}_1, \ldots, \mathbf{m}_i$ will be resampled in the i-th hybrid.

Theorem 15. *Let* PKE *be* IND-CPA *secure. Then* PKE *is* IND-SO-CPA *secure w.r.t. the class of efficiently resamplable and G-induced distribution families over $\mathcal{M}^n$ where $B(G) = \mathsf{const}$ and G is a connected* DAG.

Precisely, for any adversary $\mathcal{A}_{\mathsf{SO}}$ run in game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE}}$ there exists an $\mathsf{IND\text{-}CPA}_{\mathsf{PKE}}$ adversary $\mathcal{B}_{\mathsf{CPA}}$ with roughly the running time of $\mathcal{A}_{\mathsf{SO}}$ plus three executions of Resamp *such that*

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \leq 3 \cdot n^{B(G_\lambda)+1} \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \ .$$

Proof. We proceed in a sequence of hybrid games $\mathsf{H}_0, \mathsf{H}_1, \ldots, \mathsf{H}_n$ and switch message $\mathbf{m}_{k+1}$ from sampled to resampled in hybrid transition H_k to H_{k+1}. Hybrid H_k will return the sampled messages for all positions $[k+1, n] \cup \mathcal{I}$, but resampled messages on all positions $[k] \setminus \mathcal{I}$ where the resampling is conditioned on *every message in* $[k+1, n] \cup \mathcal{I}$. The code for CHALLENGE in given in Fig. 5, every other procedure stays as in Fig. 2.

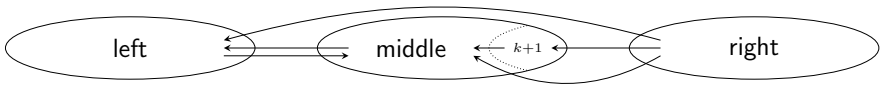

Fig. 6. Structure of G. Edges between particular sets cannot exist if there is no arrow depicted. If $\mathsf{right} \neq \emptyset$ then there is at least one edge from right to middle since G is connected. left and middle are disconnected in $G_{\overline{\mathcal{I}}}$.

Hybrid H_0 is identical to game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},0}$, whereas H_n is identical to game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},1}$, hence

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) = \mathbf{Adv}\big(\mathsf{H}_0^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_n^{\mathcal{A}_{\mathsf{SO}}}\big) \leq \sum_{k=0}^{n-1} \mathbf{Adv}\big(\mathsf{H}_k^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{k+1}^{\mathcal{A}_{\mathsf{SO}}}\big) \ . \quad (2)$$

We bound the distance between two consecutive hybrids H_k, H_{k+1} and proceed with the following lemma.

Lemma 16. *For every adversary $\mathcal{A}_{\mathsf{SO}}$ that distinguishes hybrids H_k and H_{k+1} there exists a* $\mathsf{mult\text{-}IND\text{-}CPA}$ *adversary $\mathcal{B}_{\mathsf{mult}}$ with roughly the running time of $\mathcal{A}_{\mathsf{SO}}$ plus three executions of* Resamp *such that*

$$\mathbf{Adv}\big(\mathsf{H}_k^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{k+1}^{\mathcal{A}_{\mathsf{SO}}}\big) \leq \Pr[\overline{\mathsf{Bad}}_k]^{-1} \cdot \mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) \ ,$$

where $\Pr[\overline{\mathsf{Bad}}_k]^{-1} = \sum_{i=0}^{B(G_\lambda)-1} \binom{k}{i}$ *for* $k < n-1$ *and* $\Pr[\overline{\mathsf{Bad}}_k]^{-1} = \sum_{i=0}^{B(G_\lambda)} \binom{k}{i}$ *for* $k = n-1$.

Proof Idea: We construct a $\mathsf{mult\text{-}IND\text{-}CPA}$ adversary $\mathcal{B}_{\mathsf{mult}}$ that interpolates between hybrids H_k and H_{k+1}. Ideally, $\mathcal{B}_{\mathsf{mult}}$ embeds its own challenge at position $k+1$, but might have to resample some already resampled messages in $\mathbf{m}_{[k]}$ to avoid inconsistencies. Let middle denote the connected component in $G_{[k+1]\setminus\mathcal{I}}$ that contains $\mathbf{m}_{k+1}$. Let $\mathsf{right} := [k+2, n]$, and $\mathsf{left} := \overline{(\mathsf{middle} \cup \mathsf{right})}$. Observe that it is sufficient to resample middle again to obtain consistent resampled messages. In particular, there is no need to resample any right message due to the semi-order imposed on the vertices, as a message in right does not depend on any message in $\overline{\mathsf{right}}$ (cf. Fig. 6). The reduction will guess middle to embed its $\mathsf{mult\text{-}IND\text{-}CPA}$ challenge, while it waits for all opening queries to happen to resample the left messages. Note that middle and left are disconnected in $G_{\overline{\mathcal{I}}}$, thus can be resampled independently of each other only depending on their respective neighborhood. Since right messages are fixed while resampling, it suffices to guess $N(\mathsf{middle}) \cap [k]$. Further, G is connected, i.e. $N(\mathsf{middle})$ contains at least one vertex from $\mathsf{right} = [k+2, n]$ as long as $k < n-1$. Hence, for $k < n-1$, we have $|N(\mathsf{middle}) \cap [k]| \leq B(G) - 1$.

Proof (Lemma 16). For $k \in [0, n]$ and $i \in [n]$ let $\mathtt{Open}_k(i)$ denote the event that $\mathcal{A}_{\mathsf{SO}}$ calls $\textsc{Open}(i)$ in hybrid H_k. Two arbitrary hybrids only differ in the $\textsc{Challenge}$ procedure, hence $\Pr[\mathtt{Open}_s(i)] = \Pr[\mathtt{Open}_t(i)]$ for all $s, t \in [0, n]$,

Procedure INITIALIZE
$pk \leftarrow \text{INITIALIZE}_{\mathsf{mult\text{-}IND\text{-}CPA}}(1^\lambda)$
Return pk

Procedure ENC$(\mathfrak{D}, \mathsf{Resamp}_{\mathfrak{D}})$
if $k < n-1$
  $N^* \leftarrow\!\!\$ \{V' \subseteq [k] \colon |V'| \in [0, B(G)-1]\}$
else
  $N^* \leftarrow\!\!\$ \{V' \subseteq [k] \colon |V'| \in [0, B(G)]\}$
Let middle^* denote the connected component in $G_{[k+1]\setminus N^*}$ that contains $k+1$.
$\mathbf{m}^0 \leftarrow \mathfrak{D}$
$\mathbf{m}^{1,0} \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, N^* \cup \mathsf{right})$
$\mathbf{m}^{1,1} \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, N^* \cup \mathsf{right} \cup \{k+1\})$
$\mathbf{c}_{\mathsf{middle}^*} \leftarrow \text{CHALLENGE}_{\mathsf{mult\text{-}IND\text{-}CPA}}(\mathbf{m}^{1,0}_{\mathsf{middle}^*}, \mathbf{m}^{1,1}_{\mathsf{middle}^*})$
$\mathbf{r} \leftarrow\!\!\$ \mathcal{R}^n$
$\mathbf{c}_i = \begin{cases} \mathbf{c}_i & \text{for } i \in \mathsf{middle}^* \\ \mathsf{Enc}_{pk}(\mathbf{m}^0_i; \mathbf{r}_i) & \text{else} \end{cases}$
Return $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$

Procedure OPEN(i)
if $i \in \mathsf{middle}^* \setminus \{k+1\}$
  $\mathsf{Bad} := \mathit{true}$
$\mathcal{I} := \mathcal{I} \cup \{i\}$
Return $(\mathbf{m}^0_i, \mathbf{r}_i)$

Procedure CHALLENGE
if $N^* \not\subseteq \mathcal{I}$
  $\mathsf{Bad} := \mathit{true}$
$\mathbf{m}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, \mathcal{I} \cup \mathsf{right})$
$\mathbf{m}_i = \begin{cases} \mathbf{m}^1_i & \text{for } i \in \mathsf{left} \\ \mathbf{m}^0_i & \text{else} \end{cases}$
Return $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$

Procedure FINALIZE(b')
$\text{FINALIZE}_{\mathsf{mult\text{-}IND\text{-}CPA}}(b')$

Fig. 7. $\mathcal{A}_{\mathsf{SO}}$'s game interface as provided by $\mathcal{B}_{\mathsf{mult}}$ run in game mult-IND-CPA. $\mathcal{B}_{\mathsf{mult}}$ interpolates between hybrids H_k, H_{k+1} for $k \in [0, n-1]$.

for all $i \in [n]$. Additionally, two consecutive hybrids H_k, H_{k+1} only differ in the $(k+1)$-th message returned during CHALLENGE *unless* $\mathcal{A}_{\mathsf{SO}}$ calls OPEN$(k+1)$ in game H_{k+1}. Thus, we have

$$\Pr[\mathsf{H}_k^{\mathcal{A}_{\mathsf{SO}}} \Rightarrow 1 \wedge \mathtt{Open}_k(k+1)] = \Pr[\mathsf{H}_{k+1}^{\mathcal{A}_{\mathsf{SO}}} \Rightarrow 1 \wedge \mathtt{Open}_{k+1}(k+1)]$$

and obtain

$$\mathbf{Adv}(\mathsf{H}_k^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{k+1}^{\mathcal{A}_{\mathsf{SO}}}) = \left| \Pr\left[\mathsf{H}_{k+1}^{\mathcal{A}_{\mathsf{SO}}} \Rightarrow 1 \wedge \overline{\mathtt{Open}_{k+1}(k+1)}\right] - \Pr\left[\mathsf{H}_k^{\mathcal{A}_{\mathsf{SO}}} \Rightarrow 1 \wedge \overline{\mathtt{Open}_k(k+1)}\right] \right| . \quad (3)$$

We describe $\mathcal{B}_{\mathsf{mult}}$ (cf. Fig. 7): It passes pk on to $\mathcal{A}_{\mathsf{SO}}$; obtaining $(\mathfrak{D}, \mathsf{Resamp}_{\mathfrak{D}})$, $\mathcal{B}_{\mathsf{mult}}$ makes a guess for middle (labeled middle^*) by making a guess (labeled N^*) of middle's neighborhood in $G_{[k+1]}$ and samples $\mathbf{m}^0 \leftarrow \mathfrak{D}$. $\mathcal{B}_{\mathsf{mult}}$ resamples $\mathbf{m}^{1,0}$ fixing $N^* \cup \mathsf{right}$ and resamples $\mathbf{m}^{1,1}$ fixing $N^* \cup \mathsf{right} \cup \{k+1\}$. $\mathcal{B}_{\mathsf{mult}}$ sends $(\mathbf{m}^{1,0}_{\mathsf{middle}^*}, \mathbf{m}^{1,1}_{\mathsf{middle}^*})$ to its mult-IND-CPA challenger, receives $\mathbf{c}_{\mathsf{middle}^*}$, samples fresh randomness to encrypt messages in $\overline{\mathsf{middle}^*}$ on its own and forwards $(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ to $\mathcal{A}_{\mathsf{SO}}$. $\mathcal{B}_{\mathsf{mult}}$ sets $\mathsf{Bad} := \mathit{true}$ if $\mathcal{A}_{\mathsf{SO}}$ calls OPEN(i) for some $i \in \mathsf{middle}^* \setminus \{k+1\}$ since it cannot answer those queries.[2] Other opening queries are answered honestly. On $\mathcal{A}_{\mathsf{SO}}$'s call of CHALLENGE, $\mathcal{B}_{\mathsf{mult}}$ checks if $N^* \subseteq \mathcal{I}$. If not, $\mathcal{B}_{\mathsf{mult}}$ guessed middle wrong and sets Bad to *true*. Otherwise, $\mathcal{B}_{\mathsf{mult}}$ resamples

[2] Equation (3) directly accounts for $\mathcal{A}_{\mathsf{SO}}$ calling OPEN$(k+1)$.

messages fixing those at positions $\mathcal{I} \cup \mathsf{right}$ to obtain resampled messages $\mathbf{m}^1$ and sends $\mathbf{m}_i^1$ for all left positions and $\mathbf{m}_i^0$ for all remaining positions to $\mathcal{A}_{\mathsf{SO}}$. $\mathcal{B}_{\mathsf{mult}}$ outputs whatever $\mathcal{A}_{\mathsf{SO}}$ outputs.

Assume that $\mathcal{B}_{\mathsf{mult}}$ guessed correctly, i.e. N^* is the neighborhood of middle in $G_{[k]}$. Then $\mathsf{middle}^* = \mathsf{middle}$ holds and by definition of middle, $\mathtt{Bad}$ cannot happen.

Clearly, $\mathcal{B}_{\mathsf{mult}}$ correctly simulates $\mathcal{A}_{\mathsf{SO}}$'s hybrid view in all left and right positions. Note that $\mathcal{A}_{\mathsf{SO}}$ obtains *resampled* encryptions $\mathsf{Enc}_{pk}(\mathbf{m}^{1,b}_{\mathsf{middle}})$ during Enc, but expects *sampled* encryptions $\mathsf{Enc}_{pk}(\mathbf{m}^{0}_{\mathsf{middle}})$, while receiving *sampled* $\mathbf{m}^{0}_{\mathsf{middle}}$ when calling Challenge, expecting *resampled* $\mathbf{m}_{\mathsf{middle}}$. Thus, *sampled* middle messages become *resampled* middle messages from $\mathcal{A}_{\mathsf{SO}}$'s view and vice versa.

However, we have $\mathbf{m}_{\mathsf{middle}} \equiv \mathbf{m}^0_{\mathsf{middle}}$ since $N(\mathsf{middle}) \subseteq \mathcal{I} \cup \mathsf{right}$, whereby $\mathcal{I} \cup \mathsf{right}$ is fixed when resampling $\mathbf{m}_{\mathsf{middle}}$.

For $\mathcal{B}_{\mathsf{mult}}$ run in game $\mathsf{mult\text{-}IND\text{-}CPA}_{\mathsf{PKE},0}$, $\mathcal{A}_{\mathsf{SO}}$ receives $\mathsf{Enc}_{pk}(\mathbf{m}^{1,0}_{\mathsf{middle}})$ where $\mathbf{m}^{1,0}_{\mathsf{middle}} \equiv \mathbf{m}^0_{\mathsf{middle}}$ since $N^* \cup \mathsf{right} = N \cup \mathsf{right}$ is fixed when $\mathbf{m}^{1,0}$ is resampled. Hence, *all* middle messages sent during Challenge look resampled and $\mathcal{A}_{\mathsf{SO}}$'s view is identical to hybrid H_{k+1}.

When $\mathcal{B}_{\mathsf{mult}}$ is run in $\mathsf{mult\text{-}IND\text{-}CPA}_{\mathsf{PKE},1}$, it forwards $\mathsf{Enc}_{pk}(\mathbf{m}^{1,1}_{\mathsf{middle}})$ to $\mathcal{A}_{\mathsf{SO}}$ where $\mathbf{m}^{1,1}_{\mathsf{middle}} \equiv \mathbf{m}^1_{\mathsf{middle}}$ for the same reason as for $b = 0$. In particular, we have $\mathbf{m}^0_{k+1} = \mathbf{m}^{1,1}_{k+1}$ since $\mathbf{m}^0_{k+1}$ is fixed while resampling. Consequently, each message in middle *except* the $(k+1)$-th looks resampled during Challenge and $\mathcal{A}_{\mathsf{SO}}$'s view is identical to hybrid H_k.

$\mathcal{B}_{\mathsf{mult}}$ outputs 1 in its game $\mathsf{mult\text{-}IND\text{-}CPA}$ if $\mathcal{A}_{\mathsf{SO}}$ outputs 1 in its respective hybrid and $\mathcal{A}_{\mathsf{SO}}$ does not open ciphertext $\mathbf{c}_{k+1}$ and $\mathtt{Bad}$ does not happen. We thus have

$$\begin{aligned}
&\mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) \geq \\
&\qquad \left|\Pr[\mathsf{mult\text{-}IND\text{-}CPA}^{\mathcal{B}_{\mathsf{mult}}}_{\mathsf{PKE},0} \Rightarrow 1] - \Pr[\mathsf{mult\text{-}IND\text{-}CPA}^{\mathcal{B}_{\mathsf{mult}}}_{\mathsf{PKE},1} \Rightarrow 1]\right| \\
&= \left|\Pr[\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k+1} \Rightarrow 1 \wedge \overline{\mathtt{Open}_{k+1}(k{+}1)} \wedge \overline{\mathtt{Bad}}] - \Pr[\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k} \Rightarrow 1 \wedge \overline{\mathtt{Open}_{k}(k{+}1)} \wedge \overline{\mathtt{Bad}}]\right| .
\end{aligned}$$

Since $\overline{\mathtt{Bad}}$ is independent of $\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_i \Rightarrow 1 \wedge \mathtt{Open}_i(k+1)$ for $i \in \{k, k+1\}$ we have

$$\begin{aligned}
&= \Pr[\overline{\mathtt{Bad}}] \cdot \left|\Pr[\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k+1} \Rightarrow 1 \wedge \overline{\mathtt{Open}_{k+1}(k+1)}] - \Pr[\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k} \Rightarrow 1 \wedge \overline{\mathtt{Open}_{k}(k+1)}]\right| \\
&= \Pr[\overline{\mathtt{Bad}}] \cdot \mathbf{Adv}\big(\mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k}, \mathsf{H}^{\mathcal{A}_{\mathsf{SO}}}_{k+1}\big) ,
\end{aligned}$$

by Equation (3). $\mathcal{B}_{\mathsf{mult}}$ picks N^* from a set of size $\sum_{i=0}^{B(G\lambda)-1} \binom{k}{i}$ for $k < n-1$, and of size $\sum_{i=0}^{B(G\lambda)} \binom{k}{i}$ for $k = n-1$, respectively, which proves Lemma 16. □

The remaining proof consists of tedious computations. From Equation (2) and Lemma 16 we have

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \leq \sum_{k=0}^{n-1} \Pr[\overline{\mathtt{Bad}_k}]^{-1} \cdot \mathbf{Adv}^{\mathsf{mult\text{-}IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) .$$

Let $B := B(G)$. Since $\mathcal{B}_{\mathsf{mult}}$ submits message vectors of length $|\mathsf{middle}^*| \leq k+1$ to its mult-IND-CPA challenger and by Lemma 2:

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \leq \left(\sum_{k=0}^{n-2}(k+1)\cdot\sum_{i=0}^{B-1}\binom{k}{i} + n\cdot\sum_{i=0}^{B}\binom{n-1}{i}\right)\cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{mult}}, \lambda) \ . \quad (4)$$

We upper-bound the loss in (4). Let $2 \leq B < n$.

$$\begin{aligned}
&\sum_{k=0}^{n-2}(k+1)\cdot\sum_{i=0}^{B-1}\binom{k}{i} + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&= \sum_{i=0}^{B-1}\binom{0}{i} + 2\cdot\sum_{i=0}^{B-1}\binom{1}{i} + \sum_{k=2}^{n-2}(k+1)\cdot\sum_{i=0}^{B-1}\binom{k}{i} + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&\leq 5 + \sum_{k=2}^{n-2}(k+1)\cdot\sum_{i=0}^{B-1}k^i + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&= 5 + \sum_{k=2}^{n-2}(k+1)\cdot\frac{k^B-1}{k-1} + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&= 5 + \sum_{k=2}^{n-2}\underbrace{\frac{k+1}{k-1}}_{\leq 3}\cdot(k^B-1) + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&\leq 5 + 3\cdot\sum_{k=2}^{n-2}(k^B-1) + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&= 5 + 3\cdot\sum_{k=2}^{n-2}k^B - 3\cdot(n-3) + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&= 14 - 3n + 3\cdot\sum_{k=2}^{n-2}k^B + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \\
&= 11 - 3n + 3\cdot\sum_{k=0}^{n-2}k^B + n\cdot\sum_{i=0}^{B}\binom{n-1}{i} \qquad \text{since } B \geq 1 \\
&\leq 11 - 3n + 3\cdot\sum_{k=0}^{n-2}k^B + n\cdot\sum_{i=0}^{B}n^i = 11 - 3n + 3\cdot\sum_{k=0}^{n-2}k^B + n\cdot\frac{n^{B+1}-1}{n-1} \\
&= 11 - 3n + 3\cdot\sum_{k=0}^{n-2}k^B + \underbrace{\frac{n}{n-1}}_{\leq 2}\cdot(n^{B+1}-1) \qquad \text{since } n \geq 2
\end{aligned}$$

$$\leq 9 - 3n + 3 \cdot \sum_{k=0}^{n-2} k^B + 2 \cdot n^{B+1} \leq 9 - 3n + 3 \cdot \int_0^n k^B \mathrm{d}k + 2 \cdot n^{B+1}$$

$$= 9 - 3n + 3 \cdot \frac{n^{B+1}}{B+1} + 2 \cdot n^{B+1} = 9 - 3n + \left(2 + \frac{3}{B+1}\right) \cdot n^{B+1}$$

$$\leq 9 - 3n + 3 \cdot n^{B+1} \qquad \text{since } B \geq 2$$

$$\leq 3 \cdot n^{B+1} \qquad \text{since } n \geq 3 \ .$$

Since G is connected we have $B = 0 \Leftrightarrow n = 1$, $B = 1 \Leftrightarrow n = 2$. Thus, it is easily verified that the bound holds for $(B, n) \in \{(0, 1), (1, 2)\}$ as well. □

Because Markov distributions are DAG-induced by chain graphs and the maximum border of a chain graph is at most 2 we immediately obtain a tighter version of Corollary 10 whose proof directly follows from Theorem 15.

Corollary 17. *Let* PKE *be* IND-CPA *secure. Then* PKE *is* IND-SO-CPA *secure with respect to efficiently resamplable Markov distributions over* $\mathcal{M}^n$.

In particular, for any adversary $\mathcal{A}_{\mathsf{SO}}$ *run in game* $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE}}$ *there exists an* $\mathsf{IND\text{-}CPA}_{\mathsf{PKE}}$ *adversary* $\mathcal{B}_{\mathsf{CPA}}$ *with roughly the running time of* $\mathcal{A}_{\mathsf{SO}}$ *plus three executions of* Resamp *such that*

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \leq 3 \cdot n^3 \cdot \mathbf{Adv}^{\mathsf{IND\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{B}_{\mathsf{CPA}}, \lambda) \ .$$

Applying the proof of Theorem 15 directly to the Markov case gives a slightly better bound on the loss, namely $n{\cdot}(n{+}1){\cdot}(2n{+}1)/6$, since $N(\mathsf{middle}) \cap [n{-}1] = 1$ even for the last transition H_{n-1} to H_n. Hence, the loss in Equation (4) decreases to $\sum_{k=0}^{n-1} (k+1)^2$.

Recall that the hybrids in the proof of Theorem 15 saved us a factor of n because it suffices to guess a set of size at most $B(G) - 1$ instead of $B(G)$ for $k < n - 1$ as at least one vertex of the neighborhood of middle is contained in right.

The same hybrids can be used to strengthen Theorem 8 as it suffices to guess a connected subgraph in $[k + 1]$ (instead of $[n]$) containing vertex $k + 1$.

Since G is connected, there is at least a path in $\{k + 1\} \cup \mathsf{right}$ that contains $k + 1$, i.e. at least $n - k$ connected subgraphs in $\mathsf{right} \cup \{k + 1\}$. Thus, there exist at least $n - k$ connected subgraphs in G that contain vertex $k + 1$ and are identical if restricted to $[k+1]$. Hence the probability that the reduction guesses C_{k+1} correctly can be increased from $1/S(G)$ to $(n-k)/S(G)$, bringing the loss from $\mathcal{O}(n^2) \cdot S(G)$ down to $\mathcal{O}(n \cdot \log n) \cdot S(G)$.

3.5 A Hybrid Argument for Disconnected Graphs

Let G be a graph with z' connected components. Fix any semi-order on them, e.g. ordered by the smallest vertex in each component and let $V_1, \ldots, V_{z'}$ denote the sets of vertices of the connected components of G. For $j \in [z' + 1, n]$ let $V_j := \emptyset$. We define a security game where an adversary plays the IND-SO-CPA game on a

Procedure INITIALIZE	**Procedure** OPEN(i)
$(pk, sk) \leftarrow \mathsf{Gen}(1^\lambda)$	$\mathcal{I} := \mathcal{I} \cup \{i\}$
Return pk	Return $(\mathbf{m}_i^0, \mathbf{r}_i)$
Procedure ENC($\mathfrak{D}, \mathsf{Resamp}_\mathfrak{D}$)	**Procedure** CHALLENGE
$\mathbf{m}^0 \leftarrow \mathfrak{D}$	$\mathbf{m}^1 \leftarrow \mathsf{Resamp}_\mathfrak{D}(\mathbf{m}^0, \mathcal{I})$
$\mathbf{r} \leftarrow\!\!\$\ \mathcal{R}^n$	Return $\mathbf{m}^b_{V_z}$
$\mathbf{c} = \mathsf{Enc}_{pk}(\mathbf{m}^0_{V_z}; \mathbf{r}_{V_z})$	**Procedure** FINALIZE(b')
Return $\mathbf{c}$	Return b'

Fig. 8. $\mathcal{B}_{\mathsf{G\text{-}SO}}$'s interface in game $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},b,z}$.

connected component of the graph that induced the distribution chosen by the adversary.

Definition 18. *For a public-key encryption scheme* $\mathsf{PKE} := (\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$*, a bit* b*, a family* $\mathcal{F}$ *of efficiently resamplable, G-induced distributions over* $\mathcal{M}^n$*,* $z \in [n]$ *and an adversary* $\mathcal{B}_{\mathsf{G\text{-}SO}}$ *we consider game* $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}^{\mathcal{B}_{\mathsf{G\text{-}SO}}}_{\mathsf{PKE},b,z}$ *given in Fig. 8. Run in the game,* $\mathcal{B}_{\mathsf{G\text{-}SO}}$ *calls* ENC *once right after* INITIALIZE *and submits* $\mathfrak{D} \in \mathcal{F}$ *along with a* PPT *resampling algorithm* $\mathsf{Resamp}_\mathfrak{D}$*.* $\mathcal{B}_{\mathsf{G\text{-}SO}}$ *may call* OPEN *multiple times but only for* $i \in V_z$ *and invokes* CHALLENGE *once after its last* OPEN *query before calling* FINALIZE*. We define the advantage of* $\mathcal{B}_{\mathsf{G\text{-}SO}}$ *run in* $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},b,z}$ *as*

$$\mathbf{Adv}^{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE},z}(\mathcal{B}_{\mathsf{G\text{-}SO}}, \mathfrak{D}_\lambda, \lambda) := \mathbf{Adv}(\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}^{\mathcal{B}_{\mathsf{G\text{-}SO}}}_{\mathsf{PKE},0,z}, \mathsf{G\text{-}IND\text{-}SO\text{-}CPA}^{\mathcal{B}_{\mathsf{G\text{-}SO}}}_{\mathsf{PKE},1,z}) \ .$$

PKE is $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_z$ secure w.r.t. $\mathcal{F}$ *if* $\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE},z}(\mathcal{B}_{\mathsf{G\text{-}SO}}, \mathfrak{D}_\lambda, \lambda)$ *is negligible for all* PPT *adversaries* $\mathcal{B}_{\mathsf{G\text{-}SO}}$*.* PKE is $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}$ secure w.r.t. $\mathcal{F}$ *if* PKE *is* $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_z$ *secure w.r.t.* $\mathcal{F}$ *for all* $z \in [n]$*.*

We have $\mathbf{Adv}^{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE},z}(\mathcal{B}_{\mathsf{G\text{-}SO}}, \mathfrak{D}_\lambda, \lambda) = 0$ for $z \in [z' + 1, n]$.

Theorem 19. *Let* PKE *be* $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}$ *secure w.r.t. a family* $\mathcal{F}$ *of efficiently resamplable and G-induced distributions over* $\mathcal{M}^n$*, then* PKE *is* $\mathsf{IND\text{-}SO\text{-}CPA}$ *secure w.r.t* $\mathcal{F}$*.*

Proof. Again, the main idea is that connected components can be dealt with independently. We give a hybrid argument over the connected components of G_λ using $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_z$ security for switching connected component z from sampled to resampled. See Fig. 9 for code of CHALLENGE in hybrid H_z; every other procedure stays as in $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},b}$ (cf. Fig. 2).
Note that H_0 is identical to game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},0}$, and $\mathsf{H}_{z'}$ is identical to game $\mathsf{IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},1}$. Thus

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) = \mathbf{Adv}\big(\mathsf{H}_0^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{z'}^{\mathcal{A}_{\mathsf{SO}}}\big) \leq \sum_{z=0}^{z'-1} \mathbf{Adv}\big(\mathsf{H}_z^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{z+1}^{\mathcal{A}_{\mathsf{SO}}}\big) \ .$$

Procedure CHALLENGE
$\mathbf{m}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, \mathcal{I})$
$\mathbf{m}_i = \begin{cases} \mathbf{m}_i^1 & \text{for } i \in \bigcup_{j=1}^{z} V_j \\ \mathbf{m}_i^0 & \text{else} \end{cases}$
Return $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$

Fig. 9. Hybrid H_z. The first z connected components are already resampled conditioned on opening queries, while the rest remain sampled.

We proceed with the following Lemma.

Lemma 20. *For every adversary $\mathcal{A}_{\mathsf{SO}}$ distinguishing hybrids H_z and H_{z+1} there exists an adversary $\mathcal{B}_{\mathsf{G\text{-}SO}}$ run in game $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},z+1}$ with roughly the running time plus one executions of Resamp such that*

$$\mathbf{Adv}\big(\mathsf{H}_z^{\mathcal{A}_{\mathsf{SO}}}, \mathsf{H}_{z+1}^{\mathcal{A}_{\mathsf{SO}}}\big) \le \mathbf{Adv}^{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE},z+1}(\mathcal{B}_{\mathsf{G\text{-}SO}}, \mathfrak{D}_\lambda, \lambda) \ .$$

Proof. We construct an adversary $\mathcal{B}_{\mathsf{G\text{-}SO}}$ that interpolates between hybrids H_z and H_{z+1} for $\mathcal{A}_{\mathsf{SO}}$. $\mathcal{B}_{\mathsf{G\text{-}SO}}$ proceeds as follows (cf. Fig. 10).
$\mathcal{B}_{\mathsf{G\text{-}SO}}$ forwards pk to $\mathcal{A}_{\mathsf{SO}}$. On $\mathcal{A}_{\mathsf{SO}}$'s call of ENC, $\mathcal{B}_{\mathsf{G\text{-}SO}}$ calls $\textsc{Enc}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{z+1}}$ to obtain an encryption $\mathbf{c}_{V_{z+1}}$ of messages in the component V_{z+1}. $\mathcal{B}_{\mathsf{G\text{-}SO}}$ samples messages $\mathbf{m}^0 \leftarrow \mathfrak{D}$ on its own and encrypts the messages in $\overline{V_{z+1}}$. $\mathcal{B}_{\mathsf{G\text{-}SO}}$ sends $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ to $\mathcal{A}_{\mathsf{SO}}$. $\mathcal{B}_{\mathsf{G\text{-}SO}}$ answers opening queries on its own unless they occur on V_{z+1}, where it invokes its $\textsc{Open}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{z+1}}$ oracle to answer. On CHALLENGE, $\mathcal{B}_{\mathsf{G\text{-}SO}}$ receives a challenge message vector $\mathbf{m}_{V_{z+1}}$ by calling $\textsc{Challenge}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{z+1}}$ and resamples $\mathbf{m}^1$ conditioned on $\mathcal{I}$. $\mathcal{B}_{\mathsf{G\text{-}SO}}$ returns resampled messages $\mathbf{m}^1$ on $\bigcup_{j=1}^{z} V_j$, its challenge messages $\mathbf{m}_{V_{z+1}}$ and sampled messages $\mathbf{m}^0$ for $\bigcup_{j=z+2}^{n} V_j$ to $\mathcal{A}_{\mathsf{SO}}$. $\mathcal{B}_{\mathsf{G\text{-}SO}}$ outputs whatever $\mathcal{A}_{\mathsf{SO}}$ outputs.

Obviously $\mathcal{B}_{\mathsf{G\text{-}SO}}$ simulates the hybrids correctly during ENC since it always returns encryptions of sampled messages. On $\mathcal{A}_{\mathsf{SO}}$'s call of CHALLENGE the messages in the first z connected components are already resampled while the messages in the last $n-z-1$ connected components are sampled as in hybrids H_z and H_{z+1}. When $\mathcal{B}_{\mathsf{G\text{-}SO}}$ is run in game $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},0,z+1}$, it obtains sampled messages for the $(z+1)$-th connected component; thus it runs $\mathcal{A}_{\mathsf{SO}}$ in hybrid H_z. When run in $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},1,z+1}$, $\mathcal{B}_{\mathsf{G\text{-}SO}}$ receives resampled messages for V_{z+1}; hence running $\mathcal{A}_{\mathsf{SO}}$ in hybrid H_{z+1}. Thus

$$\Pr[\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}^{\mathcal{B}_{\mathsf{G\text{-}SO}}}_{\mathsf{PKE},0,z+1} \Rightarrow 1] = \Pr[\mathsf{H}_z^{\mathcal{A}_{\mathsf{SO}}} \Rightarrow 1] \ \text{ and}$$

$$\Pr[\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}^{\mathcal{B}_{\mathsf{G\text{-}SO}}}_{\mathsf{PKE},1,z+1} \Rightarrow 1] = \Pr[\mathsf{H}_{z+1}^{\mathcal{A}_{\mathsf{SO}}} \Rightarrow 1] \ .$$

Lemma 20 follows. □

We obtain

$$\mathbf{Adv}^{\mathsf{IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE}}(\mathcal{A}_{\mathsf{SO}}, \mathfrak{D}_\lambda, \lambda) \le \sum_{z=1}^{z'} \mathbf{Adv}^{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}}_{\mathsf{PKE},z}(\mathcal{B}_{\mathsf{G\text{-}SO}}, \mathfrak{D}_\lambda, \lambda)$$

Procedure INITIALIZE
$pk \leftarrow \mathsf{Gen}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{z+1}}(1^\lambda)$
Return pk

Procedure ENC($\mathfrak{D}, \mathsf{Resamp}_{\mathfrak{D}}$)
$\mathbf{c}_{V_{z+1}} \leftarrow \text{ENC}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{z+1}}(\mathfrak{D}, \mathsf{Resamp}_{\mathfrak{D}})$
$\mathbf{m}^0 \leftarrow \mathfrak{D}$
$\mathbf{r} \leftarrow\!\!\$\ \mathcal{R}^n$
$\mathbf{c}_i = \begin{cases} \mathbf{c}_i & \text{for } i \in V_{z+1} \\ \mathsf{Enc}_{pk}(\mathbf{m}_i^0; \mathbf{r}_i) & \text{else} \end{cases}$
Return $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$

Procedure FINALIZE(b')
$\text{FINALIZE}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}}(b')$

Procedure OPEN(i)
$\mathcal{I} := \mathcal{I} \cup \{i\}$
if $i \in V_{z+1}$
  Return $\text{OPEN}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{z+1}}(i)$
else
  Return $(\mathbf{m}_i^0, \mathbf{r}_i)$

Procedure CHALLENGE
$\mathbf{m}_{V_{z+1}} \leftarrow \text{CHALLENGE}_{\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{z+1}}$
$\mathbf{m}^1 \leftarrow \mathsf{Resamp}_{\mathfrak{D}}(\mathbf{m}^0, \mathcal{I})$
$\mathbf{m}_i = \begin{cases} \mathbf{m}_i^1 & \text{for } i \in \bigcup_{j=1}^{z} V_j \\ \mathbf{m}_i & \text{for } i \in V_{z+1} \\ \mathbf{m}_i^0 & \text{else} \end{cases}$
Return $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_n)$

Fig. 10. Reduction run by $\mathcal{B}_{\mathsf{G\text{-}SO}}$ to simulate H_z (or H_{z+1}) when $\mathcal{B}_{\mathsf{G\text{-}SO}}$ is run in $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},0,z+1}$ (or $\mathsf{G\text{-}IND\text{-}SO\text{-}CPA}_{\mathsf{PKE},1,z+1}$).

and Theorem 19 follows immediately since $z' \leq n$. □

In particular, we achieve versions of Theorem 8, Corollary 14 and Theorem 15 for disconnected graphs, where

$$S(G) = \sum_{i=1}^{z'} S(C_i) \quad \text{and} \quad B(G) = \max_{i \in [z']} \{B(C_i)\}$$

for a graph G consisting of connected components $C_1, \ldots, C_{z'}$.

Moreover, for $G = ([n], \emptyset)$, G-induced distributions become product distributions, i.e. the messages are sampled independently. Hence, the positive result of [3] can be seen as a special case of Theorem 19.

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