KEM Combiners[⋆]

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Abstract. Key-encapsulation mechanisms (KEMs) are a common stepping stone for constructing public-key encryption. Secure KEMs can be built from diverse assumptions, including ones related to integer factorization, discrete logarithms, error correcting codes, or lattices. In light of the recent NIST call for post-quantum secure PKE, the zoo of KEMs that are believed to be secure continues to grow. Yet, on the question of which is the *most* secure KEM opinions are divided. While using the best candidate might actually not seem necessary to survive everyday life situations, placing a wrong bet can actually be devastating, should the employed KEM eventually turn out to be vulnerable.
We introduce KEM combiners as a way to garner trust from different KEM constructions, rather than relying on a single one: We present efficient black-box constructions that, given any set of 'ingredient' KEMs, yield a new KEM that is (CCA) secure as long as at least one of the ingredient KEMs is.
As building blocks our constructions use cryptographic hash functions and blockciphers. Some corresponding security proofs require idealized models for these primitives, others get along on standard assumptions.

Keywords: secure combiners, CCA security, practical constructions

1 Introduction

Motivation for PKE combiners. Out of the public-key encryption schemes RSA-OAEP, Cramer–Shoup, ECIES, and a scheme based on the LWE hardness assumption, which one is, security-wise, the best? This question has no clear answer, as all schemes have advantages and disadvantages. For instance, RSA-OAEP is based on the arguably best studied hardness assumption but requires a random oracle. Cramer–Shoup encryption does not require a random oracle but its security reduces 'only' to a decisional assumption (DDH). While one can give a security reduction for ECIES to a computational assumption (CDH), this reduction comes with a tightness gap much bigger than that of RSA-OAEP. On the other hand, the 'security-per-bit ratio' for elliptic curve groups is assumed

[⋆] The full version of this article can be found in the IACR eprint archive as article 2018/024.

to be much better than for RSA based schemes. Finally, the LWE scheme is the only quantum-resistant candidate, although the assumption is relatively new and arguably not yet well understood. All in all, the challenge of picking the most secure PKE scheme is arguably impossible to solve. Fortunately, the challenge can be side-stepped by using a 'PKE combiner': Instead of using only one scheme to encrypt a message, one uses all four of them, combining them in a way such that security of any implies security of their combination. Thus, when using a combiner, placing wrong bets is impossible. PKE combiners have been studied in [6,22] and we give some details on encryption combiners below.

Combiners for other cryptographic primitives. In principle, secure combiners can be studied for any cryptographic primitive. For some primitives they are easily constructed and known for quite some time. For instance, sequentially composing multiple independently keyed blockciphers to a single keyed permutation can be seen as implementing a (S)PRP combiner. PRFs can be combined by XORing their outputs into a single value. More intriguing is studying hash function combiners: Parallelly composing hash functions is a good approach if the goal is collision resistance, but pre-image resistance suffers from this. A sequential composition would be better with respect to the latter, but this again harms collision resistance. Hash function combiners that preserve both properties simultaneously exist and can be based on Feistel structures [9]. If indifferentiability from a random oracle is an additional goal, pure Feistel systems become insecure and more involved combiners are required [10,11]. Recently, also combiners for indistinguishability obfuscation have been proposed [1,8]. For an overview of combiners in cryptography we refer to [15,14].

Our target: KEM combiners. Following the contemporary KEM/DEM design principle of public-key encryption [4], in this work we study combiners for key-encapsulation mechanisms (KEMs). That is, given a set of KEMs, an unknown subset of which might be arbitrarily insecure, we investigate how they can be combined to form a single KEM that is secure if at least one ingredient KEM is. How such a combiner is constructed certainly depends on the specifics of the security goal. For instance, if CPA security shall be reached then it can be expected that combining a set of KEMs by running the encapsulation algorithms in parallel and XORing the established session keys together is sufficient. However, if CCA security is intended this construction is obviously weak.

The focus of this paper is on constructing combiners for CCA security. We propose several candidates and analyze them.[3] We stress that our focus is on practicality, i.e., the combiners we propose do not introduce much overhead and are designed such that system engineers can easily adopt them. Besides the ingredient KEMs, our combiners also mix in further cryptographic primitives like blockciphers, PRFs, or hash functions. We consider this an acceptable compromise, since they make secure constructions very efficient and arguably are not

[3] Obviously, showing *feasibility* is not a concern for KEM combiners as combiners for PKE have already been studied (see Section 1.2) and the step from PKE to KEMs is minimal.

exposed to the threats we want to hedge against. For instance, the damage that quantum computers do on AES and SHA256 are generally assumed to be limited and controllable, tightness gaps can effectively and cheaply be closed by increasing key lengths and block sizes, and their security is often easier to assume than that of number-theoretic assumptions. While, admittedly, for some of our combiners we do require strong properties of the symmetric building blocks (random oracle model, ideal cipher model, etc.), we also construct a KEM combiner that is, at a higher computational cost, secure in the standard model. In the end we offer a selection of combiners, all with specific security and efficiency features, so that for every need there is a suitable one.

1.1 Our Results

The KEM combiners treated in this paper have a parallel structure: If the number of KEMs to be combined is n, a public key of the resulting KEM consists of a vector of n public keys, one for each ingredient; likewise for secret keys. The encapsulation procedure performs n independent encapsulations, one for each combined KEM. The ciphertext of the resulting KEM is simply the concatenation of all generated ciphertexts. The session key is obtained as a function W of keys and ciphertexts (which is arguably the *core function* of the KEM combiner). A first proposal for a KEM combiner would be to use as session key the value

$$K = H(k_1, \ldots, k_n, c_1, \ldots, c_n) \ ,$$

where H is a hash function modeled as a random oracle and the pair (k_i, c_i) is the result of encapsulation under the ith ingredient KEM. A slightly more efficient combiner would be

$$K = H(k_1 \oplus \ldots \oplus k_n, c_1, \ldots, c_n) \ ,$$

where the input session keys are XOR-combined before being fed into the random oracle. On the one hand these constructions are secure, as we prove, but somewhat unfortunate is that they depend so strongly on H behaving like a random oracle: Indeed, if the second construction were to be reinterpreted as

$$K = F(k_1 \oplus \ldots \oplus k_n, c_1 \parallel \ldots \parallel c_n) \ ,$$

where now F is a (standard model) PRF, then the construction would be insecure (more precisely, we prove that there exists a PRF such that when it is used in the construction the resulting KEM is insecure). The reason for the last construction not working is that the linearity of the XOR operation allows for conducting related-key attacks on the PRF, and PRFs in general are not immune against such attacks.

Our next proposal towards a KEM combiner that is provably secure in the standard model involves thus a stronger "key-mixing component", i.e., one that is stronger than XOR. Concretely, we study the design that derives the PRF key from a chain of blockcipher invocations, each with individual key, on input the fixed value 0. We obtain

$$K = F(\pi_{k_n} \circ \ldots \circ \pi_{k_1}(0), c_1 \parallel \ldots \parallel c_n) \ ,$$

where π_k represents a blockcipher π keyed with key k. Unfortunately, also this construction is generally not secure in the standard model. Yet it is—overall—our favorite construction, for the following reason: In practice, one could instantiate F with a construction based on SHA256 (prepend the key to the message before hashing it, or use NMAC or HMAC), and π with AES. Arguably, SHA256 and AES likely behave well as PRFs and PRPs, respectively; further, in principle, SHA256 is a good candidate for a random oracle and AES is a good candidate for an ideal cipher. Our results on above combiner are as follows: While the combiner is not secure if F and π are a standard model PRF and PRP, respectively, two sufficient conditions for the KEM combiner being secure are that F is a random oracle and π a PRP *or* F is a PRF and π an ideal cipher. That is, who uses the named combiner can afford that one of the two primitives, SHA256 or AES, fails to behave like an ideal primitive. Observe that this is a clear advantage over our first two (random oracle based) combiners for which security is likely gone in the moment hash function H fails to be a random oracle.

The attentive reader might have noticed that, so far, we did not propose a KEM combiner secure in the standard model. As our final contribution we remedy this absence. In fact, by following a new approach we propose a standard-model secure KEM combiner. Concretely, if below we write $c = c_1 \parallel \ldots \parallel c_n$ for the ciphertext vector, our first standard model KEM combiner is

$$K = F(k_1, c) \oplus \ldots \oplus F(k_n, c) \ .$$

While being provably secure if F is a (standard model) PRF, the disadvantage over the earlier designs that are secure in idealized models is that this construction is less efficient, requiring n full passes over the ciphertext vector. Whether this is affordable or not depends on the particular application and the size of the KEM ciphertexts (which might be large for post-quantum KEMs).

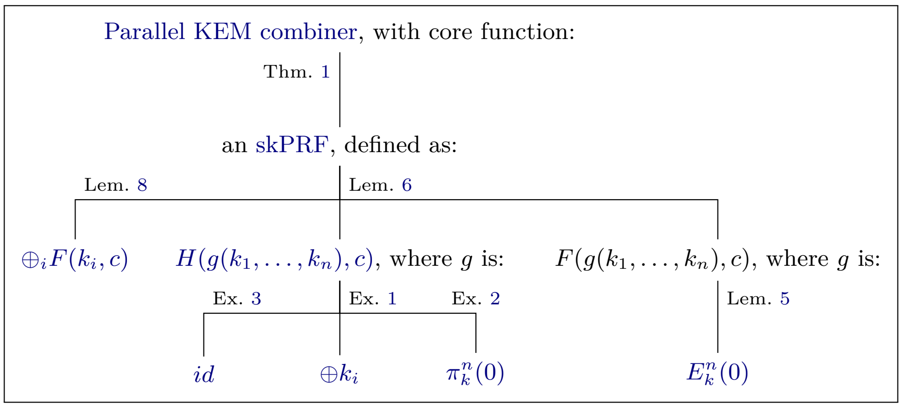
In the full version of this paper (see [13]) we give an optimized variant of above combiner where the amount of PRF-processed data is slightly smaller. Exploiting that the ciphertexts of CCA secure KEMs are non-malleable (in the sense of: If a single ciphertext bit flips the session key to which this ciphertext decapsulates is independent of the original one) we observe that the PRF invocation associated with the ith session key actually does not need to process the ith ciphertext component. More precisely, if for all i we write $c^i = c_1 \parallel \ldots \parallel c_{i-1} \parallel c_{i+1} \parallel \ldots \parallel c_n$, then also

$$K = F(k_1, c^1) \oplus \ldots \oplus F(k_n, c^n)$$

is a secure KEM combiner.

SPLIT-KEY PSEUDORANDOM FUNCTIONS Note that in all our constructions the session keys output by the KEM combiner are derived via a function of the form

$$K = W(k_1, \ldots, k_n, c) \ ,$$



Parallel KEM combiner, with core function:
Thm. 1
an skPRF, defined as:
Lem. 8
Lem. 6
$\oplus_i F(k_i, c)$
$H(g(k_1, \ldots, k_n), c)$, where g is:
$F(g(k_1, \ldots, k_n), c)$, where g is:
Ex. 3
Ex. 1
Ex. 2
Lem. 5
id
$\oplus k_i$
$\pi_k^n(0)$
$E_k^n(0)$

Fig. 1. Overview of our CCA-preserving KEM combiners for n KEMs. F denotes a PRF, H a random oracle, π a keyed permutation, and E an ideal cipher. Moreover, we assume $c = c_1 \mathinner{..} c_n$, $k = k_1 \mathinner{..} k_n$ and write $\oplus_i$ for $\oplus_{i=1}^n$. For $x \in \{\pi, E\}$ we write $x_k^n(\cdot)$ to denote $x_{k_n}(\ldots x_{k_1}(\cdot)\ldots)$. The left-most construction, $\oplus_i F(k_i, c)$, is secure in the standard model, while the remaining constructions require idealized primitives to be proven secure.

where k_i denotes the key output by the encapsulation algorithm of KEM K_i and $c = c_1 \parallel \ldots \parallel c_n$. We refer to W as *core function.* We can pinpoint a sufficient condition of the core function such that the respective KEM combiner retains CCA security of any of its ingredient KEMs: Intuitively, *split-key pseudorandomness* captures pseudorandom behavior of W as long as any of the keys $k_1, \ldots, k_n$ is uniformly distributed (and the other keys known to or controlled by the adversary).

All KEM combiners studied in this work that retain CCA security may be found in Figure 1.

1.2 Related Work

To the best of our knowledge KEM combiners have not been studied in the literature before. However, closely related, encryption combiners were considered. The idea of encrypting multiple times to strengthen security guarantees dates back to the seminal work of Shannon [21].

An immediate and well-studied solution (e.g. [5,19]) to combine various symmetric encryption schemes is to apply them in a cascade fashion where the message is encrypted using the first scheme, the resulting ciphertext then being encrypted with the second scheme, and so on. Even and Goldreich [7] showed that such a chain is at least as secure as its weakest link.Is this for CPA or CCA? I guess only CPA, but in the context of our paper it would actually be interesting to know precisely.

Focusing on combining PKE schemes and improving on prior work (see [23]) Dodis and Katz [6] gave means to employ various PKE schemes that retain CCA security of any 'ingredient' scheme.

More recently, the work of [22] gave another way to combine PKE schemes ensuring that CCA security of any ingredient PKE is passed on to the combined PKE scheme. As a first step, their approach constructs a combiner achieving merely *detectable* CCA (DCCA) security[4] if any ingredient PKE scheme is CCA secure. Secondly, a transformation from DCCA to CCA security (see [17]) is applied to strengthen the PKE combiner.

Conceptually interesting in the context of this paper is the work of [2] where the authors propose an LWE-based key exchange and integrate it into the TLS protocol suite. The goal is to make TLS future proof (against quantum computers). Thereby, they define not only two LWE-based cipher suites, but also two hybrid ones that, conservatively with respect to the security assumptions, combine the LWE techniques with better-studied cyclic group based Diffie–Hellman key exchange.

2 Preliminaries

Notation We use the following operators for assigning values to variables: The symbol '$\leftarrow$' is used to assign to a variable (on the left-hand side) a constant value (on the right-hand side), for example the output of a deterministic algorithm. Similarly, we use '$\leftarrow_{\$}$' to assign to a variable either a uniformly sampled value from a set or the output of a randomized algorithm. If $f\colon A \to B$ is a function or a deterministic algorithm we let $[f] := f(A) \subseteq B$ denote the image of A under f; if $f\colon A \to B$ is a randomized algorithm with randomness space R we correspondingly let $[f] := f(A \times R) \subseteq B$ denote the set of all its possible outputs.

Let T be an associative array (also called array, or table), and b any element. Writing '$T[\cdot] \leftarrow b$' we set $T[a]$ to b for all a. We let $[T]$ denote the space of all elements the form $T[a]$ for some a, excluding the rejection symbol $\perp$. Moreover, $[T[a, \cdot]]$ is the set of all the elements assigned to $T[a, a']$ for any value a'.

Games. Our security definitions are given in terms of *games* written in pseudocode. Within a game a (possibly) stateful *adversary* is explicitly invoked. Depending on the game, the adversary may have oracle access to specific procedures. We write $\mathcal{A}^{\mathcal{O}}$, to indicate that algorithm $\mathcal{A}$ has oracle access to $\mathcal{O}$. Within an oracle, command 'Return X' returns X to the algorithm that called the oracle.

A game terminates when a 'Stop with X' command is executed; X then serves as the output of the game. We write 'Abort' as an abbreviation for 'Stop with 0'. With '$\mathrm{G} \Rightarrow 1$' we denote the random variable (with randomness space

[4] A confidentiality notion that interpolates between CPA and CCA security. Here, an adversary is given a crippled decryption oracle that refuses to decrypt a specified set of efficiently recognizable ciphertexts. See [17].

specified by the specifics of the game G) that returns true if the output of the game is 1 and false otherwise.

In proofs that employ game hopping, lines of code that end with a comment of the form '$|\mathrm{G}_i$-G_j' (resp. '$|\mathrm{G}_i,\mathrm{G}_j$', '$|\mathrm{G}_i$') are only executed when a game in G_i–G_j (resp. G_i and G_j, G_i) is run.

Key encapsulation. A key-encapsulation mechanism (KEM) $\mathsf{K} = (\mathsf{K.gen}, \mathsf{K.enc}, \mathsf{K.dec})$ for a finite session-key space $\mathcal{K}$ is a triple of algorithms together with a public-key space $\mathcal{PK}$, a secret-key space $\mathcal{SK}$, and a ciphertext space $\mathcal{C}$. The randomized key-generation algorithm $\mathsf{K.gen}$ returns a public key $pk \in \mathcal{PK}$ and a secret key $sk \in \mathcal{SK}$. The randomized encapsulation algorithm $\mathsf{K.enc}$ takes a public key $pk \in \mathcal{PK}$ and produces a session key $k \in \mathcal{K}$ and a ciphertext $c \in \mathcal{C}$. Finally, the deterministic decapsulation algorithm $\mathsf{K.dec}$ takes a secret key $sk \in \mathcal{SK}$ and a ciphertext $c \in \mathcal{C}$, and outputs either a session key $k \in \mathcal{K}$ or the special symbol $\perp \notin \mathcal{K}$ to indicate rejection. For correctness we require that for all $(pk, sk) \in [\mathsf{K.gen}]$ and $(k, c) \in [\mathsf{K.enc}(pk)]$ we have $\mathsf{K.dec}(sk, c) = k$.

We now give a security definition for KEMs that formalizes session-key indistinguishability. For a KEM K, associate with any adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ its advantage $\mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}}(\mathcal{A})$ defined as $|\Pr[\mathrm{KIND}^0(\mathcal{A}) \Rightarrow 1] - \Pr[\mathrm{KIND}^1(\mathcal{A}) \Rightarrow 1]|$, where the games are in Figure 2. We sometimes refer to adversaries that refrain from posing queries to the Dec oracle as *passive* or *CPA*, while we refer to adversaries that pose such queries as *active* or *CCA*. Intuitively, a KEM is *CPA secure* (respectively, *CCA secure*) if all practical CPA (resp., CCA) adversaries achieve a negligible distinguishing advantage.

```
Game KIND^b(A)                              Oracle Dec(c)
00 $C^* \leftarrow \emptyset$                              08 If $c \in C^*$: Abort
01 $(pk, sk) \leftarrow_\$ \mathsf{K.gen}$                     09 $k \leftarrow \mathsf{K.dec}(sk, c)$
02 $st \leftarrow_\$ \mathcal{A}_1^{\mathrm{Dec}}(pk)$                   10 Return $k$
03 $(k^*, c^*) \leftarrow_\$ \mathsf{K.enc}(pk)$
04 $k^0 \leftarrow k^*$; $k^1 \leftarrow_\$ \mathcal{K}$
05 $C^* \leftarrow C^* \cup \{c^*\}$
06 $b' \leftarrow_\$ \mathcal{A}_2^{\mathrm{Dec}}(st, c^*, k^b)$
07 Stop with $b'$
```

Fig. 2. Security experiments KIND^b, $b \in \{0, 1\}$, modeling the session-key indistinguishability of KEM K. With st we denote internal state information of the adversary.

Pseudorandom functions. Fix a finite key space $\mathcal{K}$, an input space $\mathcal{X}$, a finite output space $\mathcal{Y}$, and a function $F \colon \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$. Towards defining what it means for F to behave pseudorandomly, associate with any adversary $\mathcal{A}$ its advantage $\mathrm{Adv}^{\mathrm{pr}}_F(\mathcal{A}) := |\Pr[\mathrm{PR}^0(\mathcal{A}) \Rightarrow 1] - \Pr[\mathrm{PR}^1(\mathcal{A}) \Rightarrow 1]|$, where the games are in Figure 3. Intuitively, F is a *pseudorandom function* (PRF) if all practical adversaries achieve a negligible advantage.

```
Game PR^b(A)                Oracle Eval(x)
00 $X \leftarrow \emptyset$                  04 If $x \in X$: Abort
01 $k \leftarrow_\$ \mathcal{K}$                 05 $X \leftarrow X \cup \{x\}$
02 $b' \leftarrow_\$ \mathcal{A}^{\mathrm{Eval}}$              06 $y \leftarrow F(k, x)$
03 Stop with $b'$              07 $y^0 \leftarrow y$; $y^1 \leftarrow_\$ \mathcal{Y}$
                            08 Return $y^b$
```

Fig. 3. Security experiments PR^b, $b \in \{0, 1\}$, modeling the pseudorandomness of function F. Line 04 and 05 implement the requirement that Eval not be queried on the same input twice.

Pseudorandom permutations. Intuitively, a pseudorandom permutation (PRP) is a bijective PRF. More precisely, if $\mathcal{K}$ is a finite key space and $\mathcal{D}$ a finite domain, then function $\pi : \mathcal{K} \times \mathcal{D} \to \mathcal{D}$ is a PRP if for all $k \in \mathcal{K}$ the partial function $\pi(k, \cdot) : \mathcal{D} \to \mathcal{D}$ is bijective and if $\pi(k, \cdot)$ behaves like a random permutation $\mathcal{D} \to \mathcal{D}$ once $k \in \mathcal{K}$ is assigned uniformly at random. A formalization of this concept would be in the spirit of Figure 3. In practice, PRPs are often implemented with blockciphers.

Random oracle model, ideal cipher model. We consider a cryptographic scheme defined with respect to a hash function $H : \mathcal{X} \to \mathcal{Y}$ in the *random oracle model* for H by replacing the scheme's internal invocations of H by calls to an oracle H that implements a uniform mapping $\mathcal{X} \to \mathcal{Y}$. In security analyses of the scheme, also the adversary gets access to this oracle. Similarly, a scheme defined with respect to a keyed permutation $\pi : \mathcal{K} \times \mathcal{D} \to \mathcal{D}$ is considered in the *ideal cipher model* for π if all computations of $\pi(\cdot, \cdot)$ in the scheme algorithms are replaced by calls to an oracle $\mathrm{E}(\cdot, \cdot)$ that implements a uniform mapping $\mathcal{K} \times \mathcal{D} \to \mathcal{D}$ such that $\mathrm{E}(k, \cdot)$ is a bijection for all k, and all computations of $\pi^{-1}(\cdot, \cdot)$ are replaced by calls to an oracle $\mathrm{D}(\cdot, \cdot)$ that implements a uniform mapping $\mathcal{K} \times \mathcal{D} \to \mathcal{D}$ such that $\mathrm{D}(k, \cdot)$ is a bijection for all k, and the partial oracles $\mathrm{E}(k, \cdot)$ and $\mathrm{D}(k, \cdot)$ are inverses of each other (again for all k). In corresponding security analyses the adversary gets access to both E and D. We write E (resp. D) to denote π (resp. π^{-1}) every time that we want to remark that π will be considered in the ideal cipher model.

3 KEM Combiners

A KEM combiner is a template that specifies how a set of existing KEMs can be joined together, possibly with the aid of further cryptographic primitives, to obtain a new KEM. In this paper we are exclusively interested in combiners that are security preserving: The resulting KEM shall be at least as secure as any of its ingredient KEMs (assuming all further primitives introduced by the combiner are secure). While for public-key encryption a serial combination process is possible and plausible (encrypt the message with the first PKE scheme, the resulting ciphertext with the second PKE scheme, and so on, for KEMs a parallel approach, where the ciphertext consists of a set of independently generated ciphertext components (one component per ingredient KEM), seems more natural. We formalize a general family of parallel combiners that are parameterized by a *core function* that derives a combined session key from a vector of session keys and a vector of ciphertexts.

Parallel KEM combiner. Let $\mathsf{K}_1, \ldots, \mathsf{K}_n$ be (ingredient) KEMs such that each $\mathsf{K}_i = (\mathsf{K.gen}_i, \mathsf{K.enc}_i, \mathsf{K.dec}_i)$ has session-key space $\mathcal{K}_i$, public-key space $\mathcal{PK}_i$, secret-key space $\mathcal{SK}_i$, and ciphertext space $\mathcal{C}_i$. Let $\mathcal{K}_* = \mathcal{K}_1 \times \ldots \times \mathcal{K}_n$ and $\mathcal{PK} = \mathcal{PK}_1 \times \ldots \times \mathcal{PK}_n$ and $\mathcal{SK} = \mathcal{SK}_1 \times \ldots \times \mathcal{SK}_n$ and $\mathcal{C} = \mathcal{C}_1 \times \ldots \times \mathcal{C}_n$. Let further $\mathcal{K}$ be an auxiliary finite session-key space. For any *core function* $W\colon \mathcal{K}_* \times \mathcal{C} \to \mathcal{K}$, the *parallel combination* $\mathsf{K} := \mathsf{K}_1 \parallel \ldots \parallel \mathsf{K}_n$ with respect to W is a KEM with session-key space $\mathcal{K}$ that consists of the algorithms $\mathsf{K.gen}, \mathsf{K.enc}, \mathsf{K.dec}$ specified in Figure 4. The combined KEM K has public-key space $\mathcal{PK}$, secret-key space $\mathcal{SK}$, and ciphertext space $\mathcal{C}$. A quick inspection of the algorithms shows that if all ingredient KEMs K_i are correct, then so is K.

```
Algo $\mathsf{K.gen}$
For $i \leftarrow 1$ to $n$:
    $(pk_i, sk_i) \leftarrow_\$ \mathsf{K.gen}_i$
$pk \leftarrow (pk_1, \ldots, pk_n)$
$sk \leftarrow (sk_1, \ldots, sk_n)$
Return $(pk, sk)$

Algo $\mathsf{K.enc}(pk)$
$(pk_1, \ldots, pk_n) \leftarrow pk$
For $i \leftarrow 1$ to $n$:
    $(k_i, c_i) \leftarrow_\$ \mathsf{K.enc}_i(pk_i)$
$c \leftarrow c_1 \mathinner{..} c_n$
$k \leftarrow W(k_1, \ldots, k_n, c)$
Return $(k, c)$

Algo $\mathsf{K.dec}(sk, c)$
$(sk_1, \ldots, sk_n) \leftarrow sk$
$c_1 \mathinner{..} c_n \leftarrow c$
For $i \leftarrow 1$ to $n$:
    $k_i \leftarrow \mathsf{K.dec}_i(sk_i, c_i)$
    If $k_i = \perp$: Return $\perp$
$k \leftarrow W(k_1, \ldots, k_n, c)$
Return $k$
```

Fig. 4. Parallel KEM combiner, defined with respect to some core function W.

The security properties of the parallel combiner depend crucially on the choice of the core function W. For instance, if W maps all inputs to one fixed session key $\bar{k} \in \mathcal{K}$, the obtained KEM does not inherit any security from the ingredient KEMs. We are thus left with finding good core functions W.

3.1 The XOR Combiner

Assume ingredient KEMs that share a common binary-string session-key space: $\mathcal{K}_1 = \ldots = \mathcal{K}_n = \{0,1\}^k$ for some k. Consider the XOR core function that, disregarding its ciphertext inputs, outputs the binary sum of the key inputs. Formally, after letting $\mathcal{K} = \{0,1\}^k$ this means $\mathcal{K}_* = \mathcal{K}^n$ and

$$W\colon\ \mathcal{K}_* \times \mathcal{C} \to \mathcal{K}\ ; \quad (k_1, \ldots, k_n, c_1 \mathinner{..} c_n) \mapsto k_1 \oplus \ldots \oplus k_n \quad .$$

On W we prove two statements: If the overall goal is to obtain a CPA-secure KEM, then W is useful, in the sense that the parallel combination of KEMs with respect to W is CPA secure if at least one of the ingredient KEMs is. However, if the overall goal is CCA security, then one weak ingredient KEM is sufficient to break any parallel combination with respect to W.

Lemma 1 (XOR combiner retains CPA security). *Let $\mathsf{K}_1, \ldots, \mathsf{K}_n$ be KEMs and let W be the XOR core function. Consider the parallel combination $\mathsf{K} = \mathsf{K}_1 \,\|\, \ldots \,\|\, \mathsf{K}_n$ with respect to W. If at least one K_i is CPA secure, then also K is CPA secure. Formally, for all indices $i \in [1 \,..\, n]$ and every adversary $\mathcal{A}$ that poses no queries to the decapsulation oracle there exists an adversary $\mathcal{B}$ such that*

$$\mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}}(\mathcal{A}) = \mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}_i}(\mathcal{B}) \ ,$$

where also $\mathcal{B}$ poses no decapsulation query and its running time is about that of $\mathcal{A}$.

Proof. From any adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ against K we construct an adversary $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ against K_i as follows. Algorithm $\mathcal{B}_1$, on input $pk_i \in \mathcal{PK}_i$, first generates the $n-1$ public keys $pk_1, \ldots, pk_{i-1}, pk_{i+1}, \ldots, pk_n$ by means of $(pk_j, sk_j) \leftarrow_{\$} \mathsf{K.gen}_j$. Then it sets $pk \leftarrow (pk_1, \ldots, pk_n)$, invokes $st \leftarrow_{\$} \mathcal{A}_1(pk)$, and outputs $st' \leftarrow (st, pk_1, \ldots, pk_{i-1}, pk_{i+1}, \ldots, pk_n)$. Algorithm $\mathcal{B}_2$, on input (st', c_i^*, k_i^*), first invokes $(k_j^*, c_j^*) \leftarrow_{\$} \mathsf{K.enc}_j(pk_j)$ for all $j \neq i$, and then sets $c^* \leftarrow c_1^* \,..\, c_i^* \,..\, c_n^*$ and $k^* \leftarrow k_1^* \oplus \ldots \oplus k_i^* \oplus \ldots \oplus k_n^*$. Finally it then invokes $b' \leftarrow_{\$} \mathcal{A}_2(st, c^*, k^*)$ and outputs b'. It is easy to see that the advantages of $\mathcal{A}$ and $\mathcal{B}$ coincide. □

Remark. Consider a CCA secure KEM (for instance from the many submissions to NIST's recent Post-Quantum initiative [20]) that is constructed by, first, taking a CPA secure KEM and then applying a Fujisaki–Okamoto-like transformation [12,16,18] to it in order to obtain a CCA secure KEM.

To combine multiple KEMs that follow the above design principle, Lemma 1 already provides a highly efficient solution that retains CCA security: To this end, one would strip away the FO-like transformation of the KEMs to be combined and apply the XOR-combiner to the various CPA secure KEMs. Eventually one would apply an FO-like transformation to the XOR-combiner.

However, besides results shedding doubts on the instantiability of FO in the presence of indistinguishability obfuscation [3], we pursue generic KEM combiners that retain CCA security independently of how the ingredient KEMs achieve their security.

While it is rather obvious that the XOR-combiner is incapable of retaining CCA security of an ingredient KEM, we formally state and prove it next.

Lemma 2 (XOR combiner does not retain CCA security). *In general, the result of parallelly combining a CCA-secure KEM with other KEMs using the XOR core function is not CCA secure.*

Formally, if $n \in \mathbb{N}$ and W is the XOR core function, then for all $1 \leq i \leq n$ there exists a KEM K_i such that for any set of $n-1$ KEMs $\mathsf{K}_1, \ldots, \mathsf{K}_{i-1}$,

$\mathsf{K}_{i+1}, \ldots, \mathsf{K}_n$ *(e.g., all of them CCA secure) there exists an efficient adversary* $\mathcal{A}$ *that poses a single decapsulation query and achieves an advantage of* $\mathrm{Adv}_{\mathsf{K}}^{\mathrm{kind}}(\mathcal{A}) = 1 - 1/|\mathcal{K}|$*, where* $\mathsf{K} = \mathsf{K}_1 \parallel \ldots \parallel \mathsf{K}_n$ *is the parallel combination of* $\mathsf{K}_1, \ldots, \mathsf{K}_n$ *with respect to* W*.*

Proof. We construct KEM K_i such that public and secret keys play no role, it has only two ciphertexts, and it establishes always the same session key: Fix any $\bar{k} \in \mathcal{K}$, let $\mathcal{C}_i = \{0, 1\}$, and let $\mathsf{K.enc}_i$ and $\mathsf{K.dec}_i$ always output $(\bar{k}, 0) \in \mathcal{K} \times \mathcal{C}_i$ and $\bar{k} \in \mathcal{K}$, respectively. Define adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ such that $\mathcal{A}_1$ does nothing and $\mathcal{A}_2$, on input of c^* and k^*, parses c^* as $c_1^* \mathinner{..} c_i^* \mathinner{..} c_n^*$ (where $c_i^* = 0$), poses a decapsulation query $k^{**} \leftarrow \mathrm{Dec}(c^{**})$ on ciphertext $c^{**} = c_1^* \mathinner{..} c_{i-1}^* 1 c_{i+1}^* \mathinner{..} c_n^*$, and outputs 1 iff $k^* = k^{**}$. It is easy to see that $\mathcal{A}$ achieves the claimed advantage. $\square$

3.2 The XOR-Then-PRF Combiner

We saw that the KEM combiner that uses the core function that simply outputs the XOR sum of the session keys fails miserably to provide security against active adversaries. The main reason is that it completely ignores the ciphertext inputs, so that the latter can be altered by an adversary without affecting the corresponding session key. As an attempt to remedy this, we next consider a core function that, using a PRF, mixes all ciphertext bits into the session key that it outputs. The PRF is keyed with the XOR sum of the input session keys and shall serve as an integrity protection on the ciphertexts.

Formally, under the same constraints on $\mathcal{K}, \mathcal{K}_1, \ldots, \mathcal{K}_n, \mathcal{K}_*$ as in Section 3.1, and assuming a (pseudorandom) function $F \colon \mathcal{K} \times \mathcal{C} \to \mathcal{K}$, the XOR-then-PRF core function W_F is defined as per

$$W_F \colon\ \mathcal{K}_* \times \mathcal{C} \to \mathcal{K}\ ; \quad (k_1, \ldots, k_n, c_1 \mathinner{..} c_n) \mapsto F(k_1 \oplus \ldots \oplus k_n, c_1 \mathinner{..} c_n) \quad .$$

Of course, to leverage on the pseudorandomness of the function F its key has to be uniform. The hope, based on the intuition that at least one of the ingredient KEMs is assumed secure and thus the corresponding session key uniform, is that the XOR sum of all session keys works fine as a PRF key. Unfortunately, as we prove next, this is not the case in general. The key insight is that the pseudorandomness definition does not capture robustness against related-key attacks: We present a KEM/PRF combination where manipulating KEM ciphertexts allows to exploit a particular structure of the PRF.[5]

Lemma 3 (XOR-then-PRF combiner does not retain CCA security). *There exist KEM/PRF configurations such that if the KEM is parallelly combined with other KEMs using the XOR-then-PRF core function, then the resulting KEM is weak against active attacks. More precisely, for all* $n \in \mathbb{N}$ *and*

[5] Note that in Lemma 6 we prove that if F behaves like a random oracle and is thus free of related-key conditions, the XOR-then-PRF core function actually does yield a secure CCA combiner.

$i \in [1 \mathinner{..} n]$ there exists a KEM K_i and a (pseudorandom) function F such that for any set of $n-1$ (arbitrarily secure) KEMs $\mathsf{K}_1, \ldots, \mathsf{K}_{i-1}, \mathsf{K}_{i+1}, \ldots, \mathsf{K}_n$ there exists an efficient adversary $\mathcal{A}$ that poses a single decapsulation query and achieves advantage $\mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}}(\mathcal{A}) = 1 - 1/|\mathcal{K}|$, where $\mathsf{K} = \mathsf{K}_1 \parallel \ldots \parallel \mathsf{K}_n$ is the parallel combination of $\mathsf{K}_1, \ldots, \mathsf{K}_n$ with respect to the XOR-then-PRF core function W_F. Function F is constructed from a function F' such that if F' is pseudorandom then so is F.

Proof. In the following we write $\mathcal{K} = \{0,1\} \times \mathcal{K}'$ (where $\mathcal{K}' = \{0,1\}^{k-1}$). We construct K_i such that public and secret keys play no role, there are only two ciphertexts, and the two ciphertexts decapsulate to different session keys: Fix any $\bar{k} \in \mathcal{K}'$, let $\mathcal{C}_i = \{0,1\}$, let $\mathsf{K.enc}_i$ always output $((0, \bar{k}), 0) \in \mathcal{K} \times \mathcal{C}_i$, and let $\mathsf{K.dec}_i$, on input ciphertext $B \in \mathcal{C}_i$, output session key $(B, \bar{k}) \in \mathcal{K}$.

We next construct a specific function F and argue that it is pseudorandom. Consider the involution $\pi \colon \mathcal{C} \to \mathcal{C}$ that flips the bit value of the ith ciphertext component, i.e.,

$$\pi(c_1 \mathinner{..} c_{i-1} \, B \, c_{i+1} \mathinner{..} c_n) = c_1 \mathinner{..} c_{i-1} \, (1 - B) \, c_{i+1} \mathinner{..} c_n \quad ,$$

and let $F' \colon \mathcal{K}' \times \mathcal{C} \to \mathcal{K}$ be a (pseudorandom) function. Construct $F \colon \mathcal{K} \times \mathcal{C} \to \mathcal{K}$ from π and F' as per

$$F((D, k'), c) = \begin{cases} F'(k', c) & \text{if } D = 0 \\ F'(k', \pi(c)) & \text{if } D = 1 \end{cases} \quad . \tag{1}$$

It is not difficult to see that if F' is pseudorandom then so is F. For completeness, we give a formal statement and proof immediately after this proof.

Consider now the following adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$: Let algorithm $\mathcal{A}_1$ do nothing, and let algorithm $\mathcal{A}_2$, on input of c^* and k^*, parse the ciphertext as $c^* = c^*_1 \mathinner{..} c^*_i \mathinner{..} c^*_n$ (where $c^*_i = 0$), pose a decapsulation query $k^{**} \leftarrow \mathrm{Dec}(c^{**})$ on ciphertext $c^{**} = c^*_1 \mathinner{..} c^*_{i-1} 1 c^*_{i+1} \mathinner{..} c^*_n$, and output 1 iff $k^* = k^{**}$.

Let us analyze the advantage of $\mathcal{A}$. For all $1 \le j \le n$, let $(d_j, k'_j) \in \mathcal{K}$ be the session keys to which ciphertext components c^*_j decapsulate. That is, the session key k to which c^* decapsulates can be computed as $k = F((d_1, k'_1) \oplus \ldots \oplus (d_n, k'_n), c^*)$, by specification of W_F. By setting $D = d_1 \oplus \ldots \oplus d_n$ and expanding F into F' and π we obtain

$$k = F'(k'_1 \oplus \ldots \oplus k'_n, c^*_1 \mathinner{..} c^*_{i-1} D c^*_{i+1} \mathinner{..} c^*_n) \quad .$$

Consider next the key k^{**} that is returned by the Dec oracle. Internally, the oracle recovers the same keys $(d_1, k'_1), \ldots, (d_n, k'_n)$ as above, with exception of d_i which is inverted. Let $D^{**} = d_1 \oplus \ldots \oplus d_n$ be the corresponding (updated) sum. We obtain

$$k^{**} = F'(k'_1 \oplus \ldots \oplus k'_n, c^*_1 \mathinner{..} c^*_{i-1} (1 - D^{**}) c^*_{i+1} \mathinner{..} c^*_n) \quad .$$

Thus, as D^{**} is the inverse of D, we have $k = k^{**}$ and adversary $\mathcal{A}$ achieves the claimed advantage. □

We now give the formal statement that, if F' is a PRF then the same holds for F as defined in (1).

Lemma 4. *Let $\mathcal{K}', \mathcal{X}, \mathcal{Y}$ be sets such that $\mathcal{K}', \mathcal{Y}$ are finite. Let $F' \colon \mathcal{K}' \times \mathcal{X} \to \mathcal{Y}$ be a function, and let $\pi \colon \mathcal{X} \to \mathcal{X}$ be any (efficient) bijection.*[6] *Let $\mathcal{K} = \{0,1\} \times \mathcal{K}'$ and define function $F \colon \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ such that*

$$F((D,k'),x) = \begin{cases} F'(k',x) & \text{if } D = 0 \\ F'(k',\pi(x)) & \text{if } D = 1 \end{cases} . \tag{2}$$

Then if F' is a PRF, the same holds for F. More precisely, for every adversary $\mathcal{A}$ there is an adversary $\mathcal{B}$ such that

$$\mathrm{Adv}^{\mathrm{pr}}_{F}(\mathcal{A}) = \mathrm{Adv}^{\mathrm{pr}}_{F'}(\mathcal{B}) \ ,$$

the running times of $\mathcal{A}$ and $\mathcal{B}$ are roughly the same, and if $\mathcal{A}$ queries its evaluation oracle q_e times then $\mathcal{B}$ queries its own evaluation oracle q_e times.

Proof. Let $\mathcal{A}$ be an adversary against the pseudorandomness of F. We build an adversary $\mathcal{B}$ against the pseudorandomness of F' as follows. $\mathcal{B}$ generates a bit D and runs $\mathcal{A}$. For every Eval query of $\mathcal{A}$ on input x, adversary $\mathcal{B}$ queries its own evaluation oracle on input x if $D = 0$, or $\pi(x)$ if $D = 1$. The output of this query is returned to $\mathcal{A}$. At the end of $\mathcal{A}$'s execution its output is returned by $\mathcal{B}$.

We argue that $\mathcal{B}$ provides a correct simulation of the pseudorandomness games to $\mathcal{A}$. First we notice that if the input values to Eval by $\mathcal{A}$ are unique, so are the input values to Eval by $\mathcal{B}$, since π is a bijection and D is constant during each run of the simulation. Conversely, any input repetition by $\mathcal{A}$ leads to an input repetition by $\mathcal{B}$, thus aborting the pseudorandomness game. If $\mathcal{B}$ is playing against the real game PR^0 for F' then it correctly computes the function F for $\mathcal{A}$ and the distribution of the output to $\mathcal{A}$ is the same as that in game PR^0 for F. Otherwise $\mathcal{B}$ receives uniform independent elements from its oracle Eval, and hence correctly simulates the game PR^1 for F to $\mathcal{A}$. This proves our statement. □

3.3 KEM Combiners from Split-Key PRFs

The two core functions for the parallel KEM combiner that we studied so far did not achieve security against active attacks. We next identify a sufficient condition that guarantees satisfactory results: If the core function is *split-key pseudorandom*, and at least one of the ingredient KEMs of the parallel combiner from Figure 4 is CCA secure, then the resulting KEM is CCA secure as well.

Split-key pseudorandom functions. We say a symmetric key primitive (syntactically) uses split keys if its key space $\mathcal{K}$ is the Cartesian product of a finite number of (sub)key spaces $\mathcal{K}_1, \ldots, \mathcal{K}_n$. In the following we study the corresponding notion of split-key pseudorandom function. In principle, such functions are just a special variant of PRFs, so that the security notion of pseudorandomness

[6] No cryptographic property is required of π, just that it can be efficiently computed. An easy example is the flip-the-first-bit function.

(see Figure 3) remains meaningful. However, we introduce *split-key pseudorandomness* as a dedicated, refined property. In brief, a split-key function has this property if it behaves like a random function if at least one component of its key is picked uniformly at random (while the other components may be known or even chosen by the adversary).

For formalizing this, fix finite key spaces $\mathcal{K}_1, \ldots, \mathcal{K}_n$, an input space $\mathcal{X}$, and a finite output space $\mathcal{Y}$. Furtehr, let $\mathcal{K} = \mathcal{K}_1 \times \ldots \times \mathcal{K}_n$ and consider a function $F\colon \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$. For each index $i \in [1 \mathinner{..} n]$, associate with an adversary $\mathcal{A}$ its advantage $\mathrm{Adv}^{\mathrm{pr}}_{F,i}(\mathcal{A}) \coloneqq |\Pr[\mathrm{PR}^0_i(\mathcal{A}) \Rightarrow 1] - \Pr[\mathrm{PR}^1_i(\mathcal{A}) \Rightarrow 1]|$, where the game is given in Figure 5. Observe that, for any index i, in the game PR^b_i, $b \in \{0,1\}$, the ith key component of F is assigned at random (in line 01), while the adversary contributes the remaining $n-1$ components on a per-query basis (see line 06). We say that F is a *split-key pseudorandom function* (skPRF) if the advantages $\mathrm{Adv}^{\mathrm{pr}}_{F,i}$ for all key indices are negligible for all practical adversaries.

```
Games $\mathrm{PR}^b_i(\mathcal{A})$
00 $X \leftarrow \emptyset$
01 $k_i \leftarrow_\$ \mathcal{K}_i$
02 $b' \leftarrow_\$ \mathcal{A}^{\mathrm{Eval}}$
03 Stop with $b'$

Oracle $\mathrm{Eval}(k', x)$
04 If $x \in X$: Abort
05 $X \leftarrow X \cup \{x\}$
06 $k_1 \mathinner{..} k_{i-1} k_{i+1} \mathinner{..} k_n \leftarrow k'$
07 $y \leftarrow F(k_1 \mathinner{..} k_i \mathinner{..} k_n, x)$
08 $y^0 \leftarrow y$; $y^1 \leftarrow_\$ \mathcal{Y}$
09 Return $y^b$
```

Fig. 5. Security game PR^b_i, $b \in \{0,1\}$, $1 \le i \le n$, modeling the split-key pseudorandomness of function F. Lines 04 and 05 implement the requirement that Eval not be queried on the same input twice.

With lines 04 and 05 we require that the oracle Eval be executed at most once on an input value x, independently on the input value k'. One could imagine a relaxed version of this requirement, where Eval accepts any non-repeating input pair (k', x), thus permitting repeated values of x in distinct queries to Eval. Most of the following proofs are however not straightforward to be adapted to the relaxed definition, and in many case this would directly lead to an insecure construction. Notice, however, that our current definition of split-key pseudorandomness for a function F still suffices to prove that F is a standard PRF.

Theorem 1. *If the core function used in the parallel composition is split-key pseudorandom, the parallel combiner yields a CCA-secure KEM if at least one of the ingredient KEMs is CCA secure.*

More precisely, for all $n, \mathsf{K}_1, \ldots, \mathsf{K}_n$, if $\mathsf{K} = \mathsf{K}_1 \parallel \ldots \parallel \mathsf{K}_n$ with core function W then for all indices i and all adversaries $\mathcal{A}$ against the key indistinguishability of K there exist adversaries $\mathcal{B}$ against the key indistinguishability of K_i and $\mathcal{C}$ against the split-key pseudorandomness of W such that

$$\mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}}(\mathcal{A}) \le 2 \cdot \left(\mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}_i}(\mathcal{B}) + \mathrm{Adv}^{\mathrm{pr}}_{W,i}(\mathcal{C})\right) \ .$$

Moreover, if adversary $\mathcal{A}$ calls at most q_d times the oracle Dec*, then adversary $\mathcal{B}$ makes at most q_d calls to the oracle* Dec*, and adversary $\mathcal{C}$ makes at most $q_d + 1$ calls to the oracle* Eval*. The running times of $\mathcal{B}$ and $\mathcal{C}$ are roughly the same as that of $\mathcal{A}$.*

Proof sketch. The proof constitutes of a sequence of games interpolating between games G_0 and G_4. Noting that the KEMs we consider are perfectly correct, those two games correspond respectively to games KIND^0 and KIND^1 for the KEM $\mathsf{K} = \mathsf{K}_1 \| \ldots \| \mathsf{K}_n$. Code detailing the games involved is in Figure 7 and the main differences between consecutive games are explained in Figure 6. In a nutshell, we proceed as follows: In game G_1 we replace the key k_i^* output by $(k_i^*, c_i^*) \leftarrow_\$ \mathsf{K.enc}_i(pk_i)$ by a uniform key. As K_i is CCA secure this modification is oblivious to $\mathcal{A}$. As a second step, we replace the real challenge session key k^* as obtained via $k^* \leftarrow W(k_1^*, \ldots, k_n^*, c_1^* \mathinner{..} c_n^*)$ with a uniform session key in game G_2. Since the core function W is split-key pseudorandom and k_i^* is uniform, this step is oblivious to $\mathcal{A}$ as well. However—for technical reasons within the reduction—replacing the challenge session key will introduce an artifact to the decapsulation procedure: queries of the form $\text{Dec}(\ldots, c_i^*, \ldots)$ will not be processed using W but answered with uniform session keys. In the transition to game G_3 we employ the split-key pseudorandomness of W again to remove the artifact from the decapsulation oracle. Eventually, in game G_4 we undo our first modification and replace the currently uniform key k_i^* with the actual key obtained by running $\mathsf{K.enc}_i(pk_i)$. Still, the challenge session key k^* remains uniform. Again, the CCA security of K_i ensures that $\mathcal{A}$ will not detect the modification.

We proceed with a detailed proof.

Game	k_i^*	k^*	$\text{Dec}(\ldots, c_i^*, \ldots)$	Δ
G_0 ($\equiv \text{KIND}^0$)	real	real	real	
				$\text{Adv}^{\text{kind}}_{\mathsf{K}_i}$
G_1	random	real	real	
				$\text{Adv}^{\text{pr}}_{W,i}$
G_2	random	random	random	
				$\text{Adv}^{\text{pr}}_{W,i}$
G_3	random	random	real	
				$\text{Adv}^{\text{kind}}_{\mathsf{K}_i}$
G_4 ($\equiv \text{KIND}^1$)	real	random	real	

Fig. 6. Overview of the proof of Theorem 1. We have $(k_i^*, c_i^*) \leftarrow_\$ \mathsf{K.enc}_i(pk_i)$. Furthermore, $k^* \leftarrow W(k_1^*, \ldots, k_n^*, c_1^* \mathinner{..} c_n^*)$ denotes the challenge session key given to $\mathcal{A}_2$ along with $c_1^* \mathinner{..} c_n^*$.

Proof (Theorem 1). Let $\mathcal{A}$ denote an adversary attacking the CCA security of the KEM K that issues at most q_d queries to the decapsulation oracle. We proceed with detailed descriptions of the games (see Figure 7) used in our proof.

Game G_0 The KIND^0 game instantiated with the KEM K as given in Figure 4. Beyond that we made merely syntactical changes: In line 00 a set C_i^* and an

```
Games G_0 to G_4
00 $C^*, C_i^* \leftarrow \emptyset$; $L[\cdot] \leftarrow \bot$
01 For $j \leftarrow 1$ to $n$:
02    $(pk_j, sk_j) \leftarrow_\$ \mathsf{K.gen}_j$
03 $pk \leftarrow (pk_1, \dots, pk_n)$
04 $st \leftarrow_\$ \mathcal{A}_1^{\mathrm{Dec}}(pk)$
05 For $j \leftarrow 1$ to $n$:
06    $(k_j^*, c_j^*) \leftarrow_\$ \mathsf{K.enc}_j(pk_j)$
07 $k_i^* \leftarrow_\$ \mathcal{K}_i$                          | G_1-G_3
08 $c^* \leftarrow c_1^* \mathinner{..} c_n^*$
09 $k^* \leftarrow W(k_1^*, \dots, k_n^*, c^*)$
10 $k^* \leftarrow_\$ \mathcal{K}$                            | G_2-G_4
11 $C^* \leftarrow C^* \cup \{c^*\}$; $C_i^* \leftarrow C_i^* \cup \{c_i^*\}$
12 $b' \leftarrow_\$ \mathcal{A}_2^{\mathrm{Dec}}(st, c^*, k^*)$
13 Stop with $b'$
```

```
Oracle Dec(c)
14 If $c \in C^*$: Abort
15 If $L[c] \neq \bot$: Return $L[c]$
16 $c_1 \mathinner{..} c_n \leftarrow c$
17 For $j \in [1 \mathinner{..} n] \setminus \{i\}$:
18    $k_j \leftarrow \mathsf{K.dec}_j(sk_j, c_j)$
19    If $k_j = \bot$: Return $\bot$
20 If $c_i \in C_i^*$:
21    $k_i \leftarrow k_i^*$
22 Else:
23    $k_i \leftarrow \mathsf{K.dec}_i(sk_i, c_i)$
24    If $k_i = \bot$: Return $\bot$
25 $L[c] \leftarrow W(k_1, \dots, k_n, c)$
26 If $c_i \in C_i^*$: $L[c] \leftarrow_\$ \mathcal{K}$    | G_2
27 Return $L[c]$
```

Fig. 7. Games G_0–G_4 as used in the proof of Theorem 1. Note that i is implicitly a parameter of all games above.

array L are initialized as empty. In line 15 we check if the adversary has already queried the oracle for the same input and we return the same output. Lines 20 and 21 are added such that, instead of using sk_i to decapsulate c_i^*, the key k_i^* is used. Note that if line 21 is executed then key k_i^* is already defined, since $C_i^* \neq \emptyset$.

Claim 1. $\Pr[\mathrm{KIND}^0 \Rightarrow 1] = \Pr[G_0 \Rightarrow 1]$.

This follows immediately from the correctness of K_i and the fact that the decapsulation algorithm is deterministic.

Game G_1 Line 07 is added to replace the key k_i^* with a uniform key from $\mathcal{K}_i$.

Claim 2. There is an adversary $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ against session-key indistinguishability of K_i (see Figure 8) that issues at most q_d decapsulation queries such that

$$|\Pr[G_0 \Rightarrow 1] - \Pr[G_1 \Rightarrow 1]| \leq \mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}_i}(\mathcal{B}) \ ,$$

and the running time of $\mathcal{B}$ is roughly the running time of $\mathcal{A}$.

Proof. We construct $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ as given in Figure 8: Adversary $\mathcal{B}_1$ gets pk_i as input, and runs $(pk_j, sk_j) \leftarrow_\$ \mathsf{K.gen}_j$ for all $j \in [1 \mathinner{..} n] \setminus \{i\}$ to instantiate the other KEMs (see lines 01–03). To answer the decapsulation queries of $\mathcal{A}_1$, $\mathcal{B}_1$ decapsulates all c_i for $j \neq i$ using sk_j (lines 16–18) and queries its own decapsulation oracle to decapsulate c_i (lines 21–23).

Adversary $\mathcal{B}_2$, run on the challenge (c_i^*, k_i^*), executes $(k_j^*, c_j^*) \leftarrow_\$ \mathsf{K.enc}_j$ for $j \neq i$ on its own (lines 07, 08). Then it computes the challenge session key $k^* \leftarrow W(k_1^*, \dots, k_n^*, c_1^*, \dots, c_n^*)$ (line 10) and runs $\mathcal{A}_2$ on $(c_1^*, \dots, c_n^*, k^*)$ (line 12). Decryption queries are answered as in phase one unless $\mathcal{B}_2$ has to decapsulate c_i^*

```
**Adversary** $\mathcal{B}_1^{\mathrm{Dec}}(pk_i)$
00 $C^*, C_i^* \leftarrow \emptyset$
01 For $j \in [1\,..\,n] \setminus \{i\}$:
02     $(pk_j, sk_j) \leftarrow_\$ \mathsf{K.gen}_j$
03 $pk \leftarrow (pk_1, \ldots, pk_n)$
04 $st \leftarrow_\$ \mathcal{A}_1^{\mathrm{Dec}}(pk)$
05 $st' \leftarrow (st, pk, sk_1, \ldots, sk_{i-1}, sk_{i+1}, \ldots, sk_n)$
06 Return $st'$

**Adversary** $\mathcal{B}_2^{\mathrm{Dec}}(st', c_i^*, k_i^*)$
07 For $j \in [1\,..\,n] \setminus \{i\}$:
08     $(k_j^*, c_j^*) \leftarrow_\$ \mathsf{K.enc}_j(pk_j)$
09 $c^* \leftarrow c_1^* \,..\, c_n^*$
10 $k^* \leftarrow W(k_1^*, \ldots, k_n^*, c^*)$
11 $C^* \leftarrow C^* \cup \{c^*\}$; $C_i^* \leftarrow C_i^* \cup \{c_i^*\}$
12 $b' \leftarrow_\$ \mathcal{A}_2^{\mathrm{Dec}}(st, c^*, k^*)$
13 Stop with $b'$

If $\mathcal{A}$ calls Dec($c$):
14 If $c \in C^*$: Abort
15 $c_1 \,..\, c_n \leftarrow c$
16 For $j \in [1\,..\,n] \setminus \{i\}$:
17     $k_j \leftarrow \mathsf{K.dec}_j(sk_j, c_j)$
18     If $k_j = \bot$: Return $\bot$
19 If $c_i \in C_i^*$:
20     $k_i \leftarrow k_i^*$
21 Else:
22     $k_i \leftarrow \mathrm{Dec}(c_i)$
23     If $k_i = \bot$: Return $\bot$
24 $k \leftarrow W(k_1, \ldots, k_n, c)$
25 Return $k$
```

Fig. 8. Adversary $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ against session-key indistinguishability of K_i from adversary $(\mathcal{A}_1, \mathcal{A}_2)$ against session-key indistinguishability of K.

where it uses k_i^* instead (lines 19, 20). At the end $\mathcal{B}_2$ relays $\mathcal{A}_2$'s output and halts (line 13).

ANALYSIS Games G_0 and G_1 only differ on the key k_i^* used to compute k^* for $\mathcal{A}_2$, and, consequently, when answering $\mathcal{A}_2$'s decapsulation queries involving c_i^*. If $\mathcal{B}$ is run by the game KIND^0, that is, key k_i^* is a real key output of $\mathsf{K.enc}_i$, then $\mathcal{B}$ perfectly emulates game G_0. Otherwise, if $\mathcal{B}$ is run by the game KIND^1, and thus the key k_i^* is uniform, then $\mathcal{B}$ emulates G_1. Hence

$$\Pr[\mathrm{G}_0 \Rightarrow 1] = \Pr[\mathrm{KIND}^0 \Rightarrow 1]$$

and

$$\Pr[\mathrm{G}_1 \Rightarrow 1] = \Pr[\mathrm{KIND}^1 \Rightarrow 1] \ .$$

Lastly we observe that $\mathcal{B}$ issues at most as many decapsulation queries as $\mathcal{A}$. Our claim follows. □

Game G_2 We add line 10 and line 26. Thus, whenever W is evaluated on a ciphertext whose ith component is c_i^* (that is, either when computing the challenge session key k^* or when answering decapsulation queries involving c_i^* as the ith ciphertext component) the output is overwritten with a uniform value from $\mathcal{Y}$.

Claim 3. There exists an adversary $\mathcal{C}$ against the split-key pseudorandomness security of W that issues at most $q_d + 1$ evaluation queries such that

$$|\Pr[\mathrm{G}_1 \Rightarrow 1] - \Pr[\mathrm{G}_2 \Rightarrow 1]| \leq \mathrm{Adv}_{W,i}^{\mathrm{pr}}(\mathcal{C}) \ ,$$

and the running time of $\mathcal{C}$ is roughly the running time of $\mathcal{A}$.

Proof. We construct an adversary $\mathcal{C}$ that breaks the split-key pseudorandomness of W on the ith key if $\mathcal{A}$ distinguishes between games G_1 and G_2.

Adversary $\mathcal{C}$ runs $\mathsf{K.gen}_j$ for all $j \in [1..n]$ to instantiate all KEMs (see lines 01–03). Then for each KEM K_j it generates a pair key-ciphertext (k_j^*, c_j^*) (lines 05 and 06). All ciphertexts, and all the keys k_j^* for $j \neq i$, are collected and used as input for a call to Eval to generate $\mathcal{A}_2$'s challenge (lines 07–09). To answer the decapsulation queries of $\mathcal{A}$ on input $c_1 .. c_n$, the adversary keeps track of previous decapsulation queries and returns the same result for two queries with the same input (line 14). $\mathcal{C}$ uses the secret keys it generated to decapsulate all ciphertext components c_j for $j \neq i$ (lines 16–18). The same procedure is used to decapsulate c_i if $c_i \neq c_i^*$; otherwise it queries its own decapsulation oracle (lines 19–25).

```
Adversary $\mathcal{C}^{\mathrm{Eval}}$
00 $C^*, C_i^* \leftarrow \emptyset$; $L[\cdot] \leftarrow \perp$
01 For $j \leftarrow 1$ to $n$:
02     $(pk_j, sk_j) \leftarrow_\$ \mathsf{K.gen}_j$
03 $pk \leftarrow (pk_1, \ldots, pk_n)$
04 $st \leftarrow_\$ \mathcal{A}_1^{\mathrm{Dec}}(pk)$
05 For $j \leftarrow 1$ to $n$:
06     $(k_j^*, c_j^*) \leftarrow_\$ \mathsf{K.enc}_j(pk_j)$
07 $k'^* \leftarrow k_1^* .. k_{i-1}^* k_{i+1}^* .. k_n^*$
08 $c^* \leftarrow (c_1^*, \ldots, c_n^*)$
09 $k^* \leftarrow \mathrm{Eval}(k'^*, c^*)$
10 $C^* \leftarrow C^* \cup \{c^*\}$; $C_i^* \leftarrow C_i^* \cup \{c_i^*\}$
11 $b' \leftarrow_\$ \mathcal{A}_2^{\mathrm{Dec}}(st, c^*, k^*)$
12 Stop with $b'$

If $\mathcal{A}$ calls Dec($c$):
13 If $c \in C^*$: Abort
14 If $L[c] \neq \perp$: Return $L[c]$
15 $(c_1, \ldots, c_n) \leftarrow c$
16 For $j \in [1..n] \setminus \{i\}$:
17     $k_j \leftarrow \mathsf{K.dec}_j(sk_j, c_j)$
18     If $k_j = \perp$: Return $\perp$
19 If $c_i \in C_i^*$:
20     $k' \leftarrow k_1 .. k_{i-1} k_{i+1} .. k_n$
21     $L[c] \leftarrow \mathrm{Eval}(k', c)$
22 Else:
23     $k_i \leftarrow \mathsf{K.dec}_i(sk_i, c_i)$
24     If $k_i = \perp$: Return $\perp$
25     $L[c] \leftarrow W(k_1 .. k_i .. k_n, c)$
26 Return $L[c]$
```

Fig. 9. Adversary $\mathcal{C}$ against multi-key pseudorandomness of F.

Analysis First we note that by the conditions in lines 13 and 14 in Figure 9 all calls to Eval by $\mathcal{C}^b$ have different input and thus we can always use Eval to simulate W.

Observe that when $\mathcal{C}$ plays against PR_i^0 we are implicitly setting k_i^* as the key internally generated by PR_i^0. Hence $\mathcal{C}$ correctly simulates game G_1 to $\mathcal{A}$. Otherwise when $\mathcal{C}$ plays against PR_i^1 the oracle Eval consistently outputs random elements in $\mathcal{K}$. Thus $\mathcal{C}$ correctly simulates game G_2 to $\mathcal{A}$. Therefore

$$\Pr[G_1 \Rightarrow 1] = \Pr[\mathrm{PR}_i^0 \Rightarrow 1]$$

and

$$\Pr[G_2 \Rightarrow 1] = \Pr[\mathrm{PR}_i^1 \Rightarrow 1]| \ .$$

Thus

$$|\Pr[G_1 \Rightarrow 1] - \Pr[G_2 \Rightarrow 1]| \leq \mathrm{Adv}_{W,i}^{\mathrm{pr}}(\mathcal{C}) \ .$$

We count the number of Eval queries by $\mathcal{C}$. From the definition of $\mathcal{C}$ we see that the oracle Eval is called once to generate the challenge. Further, for each Dec query by $\mathcal{A}$, $\mathcal{C}$ queries Eval at most once. $\square$

Game G_3 We remove lines 26 to undo the modifications of the Dec oracle introduced in game G_2. Thus, during decapsulation, whenever W is evaluated on a ciphertext whose ith component is c_i^* the output is computed evaluating the function W on the decapsulated keys instead of returning a uniform input.

Claim 4. There exists an adversary $\mathcal{C}'$ against the split-key pseudorandomness security of W that issues at most q_d evaluation queries such that

$$|\Pr[G_2 \Rightarrow 1] - \Pr[G_3 \Rightarrow 1]| \le \mathrm{Adv}^{\mathrm{pr}}_{W,i}(\mathcal{C}') \ ,$$

and the running time of $\mathcal{C}'$ is roughly the running time of $\mathcal{A}$.

Proof. Adversary $\mathcal{C}'$ is essentially the same as adversary $\mathcal{C}$ in Figure 9, with the exception that we replace line 09 with the generation of a uniform session key ($k^* \leftarrow_\$ \mathcal{K}$). The proof analysis is the same as in Claim 3. Notice that since this time the challenge session key is uniform, $\mathcal{C}'$ calls Eval just q_d times instead of $q_d + 1$. $\square$

Note that, currently, the only difference from game G_1 is the addition of line 10, i.e., the challenge session key k^* is uniform.

Game G_4 Line 07 is removed to undo the modification introduced in game G_1. That is, we replace the uniform key k_i^* with a real key output by $\mathsf{K.enc}_i(pk_i)$.

Claim 5. There exists an adversary $\mathcal{B}' = (\mathcal{B}_1', \mathcal{B}_2')$ against the session-key indistinguishability of K_i that issues at most q_d decapsulation queries such that

$$|\Pr[G_3 \Rightarrow 1] - \Pr[G_4 \Rightarrow 1]| \le \mathrm{Adv}^{\mathrm{kind}}_{\mathsf{K}_i}(\mathcal{B}') \ ,$$

and the running time of $\mathcal{B}'$ is roughly the running time of $\mathcal{A}$.

Proof. Adversary $\mathcal{B}'$ is the same as adversary $\mathcal{B}$ in Figure 8, with the exception that we replace line 10 with the generation of a uniform session key ($k^* \leftarrow_\$ \mathcal{K}$). The proof analysis is the same as in Claim 2. $\square$

Claim 6. $\Pr[G_4 \Rightarrow 1] = \Pr[\mathrm{KIND}^1 \Rightarrow 1]$.

This follows immediately from the correctness of K_i and the fact that the decapsulation algorithm is deterministic.

The proof of the main statement follows from collecting the statements from Claims 1 to 6. $\square$

4 Split-Key PRFs in Idealized Models

In the previous section we have shown that if the core function of the parallel combiner is split-key pseudorandom, then said combiner preserves CCA security of any of its ingredient KEMs. It remains to present explicit, practical constructions of skPRFs.

In our first approach we proceed as follows: Given some keys $k_1, \ldots, k_n$ and some input x, we mingle together the keys to build a new key k for some (single-key) pseudorandom function F. The output of our candidate skPRF is obtained evaluating $F(k, x)$. In this section we consider variations on how to compute the PRF key k, along with formal proofs for the security of the corresponding candidate skPRFs.

Considering our parallel combiner with such skPRF, evaluating a session key becomes relatively efficient compared to the unavoidable cost of running n distinct encapsulations. Alas, the security of the constructions in this section necessitates some idealized building block, that is, a random oracle or an ideal cipher.

We attempt to abate this drawback by analyzing the following construction form different angles:

$$W(k_1, \ldots, k_n, x) \coloneqq F(\pi(k_n, \pi(\ldots \pi(k_1, 0) \ldots)), x) \ , \tag{3}$$

where F is a pseudorandom function and π is a pseudorandom permutation. Specifically, we show that W is an skPRF if π is modeled as an ideal cipher (Lemma 5) or F is modeled as a random oracle (Lemma 6 in combination with Example 2).

This statement might be interesting in practice: When implementing such construction the real world, F could reasonably be fixed to SHA-2 (prepending the key), while AES could reasonably be chosen as π. Both primitives are believed to possess good cryptographic properties, arguably so to behave as idealized primitives. Moreover, there is no indication to assume that if one primitive failed to behave 'ideally', then the other would be confronted with the same problem.

In Section 4.1 we prove that the construction above is secure in the ideal cipher model. In Section 4.2 we give some secure constructions in the case that F is modeled as a random oracle.

4.1 Split-Key PRFs in the Ideal Cipher Model

Here we consider constructions of skPRFs where the key-mixing step is conducted in the ideal cipher model followed by a (standard model) PRF evaluation.

Before stating the main result of this section we introduce two additional security notions for keyed functions. The first one is a natural extension of pseudorandomness, whereby an adversary is given access to *multiple instances* of a keyed function (under uniform keys) or truly random functions.

Multi-instance pseudorandomness See Figure 10 for the security game that defines the multi-instance pseudorandomness of F. For any adversary $\mathcal{A}$ and number of instances n we define its advantage $\mathrm{Adv}^{\mathrm{mipr}}_{F,n}(\mathcal{A}) := |\Pr[\mathrm{MIPR}^0(\mathcal{A}) \Rightarrow 1] - \Pr[\mathrm{MIPR}^1(\mathcal{A}) \Rightarrow 1]|$. Intuitively, F is *multi-instance pseudorandom* if all practical adversary achieve a negligible advantage.

```
Game MIPR^b(A)                        Oracle Eval(i, x)
00 $X_1, \ldots, X_n \leftarrow \emptyset$          04 If $x \in X_i$: Abort
01 $k_1, \ldots, k_n \leftarrow_\$ \mathcal{K}$        05 $X_i \leftarrow X_i \cup \{x\}$
02 $b' \leftarrow_\$ \mathcal{A}^{\mathrm{Eval}}$              06 $y \leftarrow F(k_i, x)$
03 Stop with $b'$                      07 $y^0 \leftarrow y$; $y^1 \leftarrow_\$ \mathcal{Y}$
                                      08 Return $y^b$
```

Fig. 10. Security experiments MIPR^b, $b \in \{0, 1\}$, modeling multi-instance pseudorandomness of F for n instances.

While one usually considers indistinguishability between outputs of a pseudorandom functions and uniform elements, key inextractability requires instead that the PRF key be hidden from any efficient adversary. We give a formalization of the latter property in the multi-instance setting next.

Multi-instance key inextractability Next we introduce multi-instance key inextractability for a keyed function F. To this end, consider the game MIKI given in Figure 11. To any adversary $\mathcal{A}$ and any number of instances n we associate its advantage $\mathrm{Adv}^{\mathrm{miki}}_{F,n}(\mathcal{A}) := \Pr[\mathrm{MIKI}(\mathcal{A}) \Rightarrow 1]$. Intuitively, F satisfies *multi-instance key inextractability* if all practical adversaries achieve a negligible advantage.

```
Game MIKI(A)                      Oracle Eval(i, x)         Oracle Check(k)
00 $k_1, \ldots, k_n \leftarrow_\$ \mathcal{K}$    03 $y \leftarrow F(k_i, x)$        05 If $k \in \{k_1, \ldots, k_n\}$:
01 Run $\mathcal{A}^{\mathrm{Eval,Check}}$           04 Return $y$              06     Stop with 1
02 Stop with 0
```

Fig. 11. Security experiment MIKI modeling multi-instance key inextractability of F for n instances.

Lemma 5. *Let $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{Y}$ be finite sets, $\mathcal{X}$ be a set and n a positive integer. Let $F\colon \mathcal{H} \times \mathcal{X} \to \mathcal{Y}$, $E\colon \mathcal{K} \times \mathcal{H} \to \mathcal{H}$, and $D\colon \mathcal{K} \times \mathcal{H} \to \mathcal{H}$ be functions such that for all $k \in \mathcal{K}$ the function $E(k, \cdot)$ is invertible with inverse $D(k, \cdot)$. Consider the function W defined by:*

$$W\colon \mathcal{K}^n \times \mathcal{X} \to \mathcal{Y} \ , \quad W(k_1, \ldots, k_n, x) := F(E(k_n, E(\ldots E(k_1, 0) \ldots)), x) \ .$$

If the function F is pseudorandom then the function W is split-key pseudorandom in the ideal cipher model.

More precisely, suppose that E is modeled as an ideal cipher with inverse D. Then for any $i \in [1 \mathinner{..} n]$ and for any adversary $\mathcal{A}$ against the split-key pseudorandomness of W there exists an adversary $\mathcal{B}$ against the multi-instance key inextractability of F and an adversary $\mathcal{C}$ against the multi-instance pseudorandomness of F such that:

$$\mathrm{Adv}^{\mathrm{pr}}_{W,i}(\mathcal{A}) \leq \frac{Q + nq_e}{|\mathcal{K}| - n} + 6 \cdot \frac{(Q + 2nq_e)^2}{|\mathcal{H}| - 2Q - 2nq_e} + \mathrm{Adv}^{\mathrm{miki}}_{F,q_e}(\mathcal{B}) + \mathrm{Adv}^{\mathrm{mipr}}_{F,q_e}(\mathcal{C}) \ ,$$

where q_e (resp. Q) is the maximum number of calls by $\mathcal{A}$ to the oracle Eval *(resp. to the ideal cipher or its inverse). Moreover, $\mathcal{B}$ calls at most q_e (resp. $2Q + nq_e$) times the oracle* Eval *(resp.* Check*), and $\mathcal{C}$ calls at most q_e times the oracle* Eval*. The running times of $\mathcal{B}$ and $\mathcal{C}$ are roughly the same as that of $\mathcal{A}$.*

Proof sketch. The proof consists of a sequence of games interpolating between the games PR^0_i and PR^1_i for any $i \in [1 \mathinner{..} n]$. Our final goal is to make the PRF keys used in Eval as input to F uniform, and then employ the PRF security of F. To achieve this we show that, except with a small probability, the adversary cannot manipulate the game to use anything but independent, uniformly generated values as key input to F.

The PRF keys are sequences of the form $h = E(k_n, E(\ldots E(k_1, 0) \ldots))$ for some keys $k_1 \mathinner{..} k_n$. We fix an index i: The key k_i is uniformly generated by the pseudorandomness game, and the remaining keys are chosen by the adversary on each query to Eval. The proof can be conceptually divided into two parts. Initially (games G_0–G_3) we work on the first part of the sequence, namely $h' = E(k_i, E(\ldots E(k_1, 0) \ldots))$. Here we build towards a game in which all elements h' that are generated from different key vectors $k_1 \mathinner{..} k_{i-1}$ are independent uniform values. In the next games (games G_4–G_9) we work on the second part of the sequence, namely $h = E(k_n, E(\ldots E(k_{i+1}, h') \ldots))$. Again, we show that all elements h are independent and uniform, assuming independent uniform values h'.

We describe now each single game hop. We start from game G_0, equivalent to the real game PR^0_i, and we proceed as follows. Game G_1 aborts if the key k_i is directly used as input by the adversary in one of its oracle queries. In game G_2 the output of E under the uniform key k_i is precomputed and stored in a list R, which is then used by Eval. Game G_3 aborts when, in a query to Eval, the adversary triggers an evaluation of $E(k_{i-1}, E(\ldots E(k_1, 0) \ldots))$ that gives the same output as one of a previous evaluations using a different key vector. At this point we want to argue that an adversary sequentially evaluating $n - i$ times the ideal cipher under know keys but uniform initial input still cannot obtain anything but a(n almost) uniform output. This will be achieved by uniformly pre-generating the enciphering output used to evaluate the sequences $E(k_n, E(\ldots E(k_{i+1}, h') \ldots))$. These elements are precomputed in game G_4 and stored in a list R, but not yet used. In game G_5 the elements stored in R are removed from the range of the ideal cipher. In game G_6, the oracle Eval uses the values in R to sample the ideal cipher. Since this might not always be possible, the oracle Eval resumes standard sampling if any value to be sampled has already been set in E or D. The next game makes a step forward to guarantee that the previous condition does not occur: If the two oracles E and D have never been queried with input any value that is used as key to the PRF F, then the game aborts if any element stored in R (but not used as a PRF key) is queried to E or D. All previous steps have only involved information-theoretical arguments. In game G_8 we disjoin our simulated ideal cipher from the PRF keys. This requires many small changes to the game structure, but eventually the price paid to switch from game G_7 is the advantage in breaking multi-instance key inextractability of the PRF, i.e., to recover one of the PRF keys from the PRF output. At this point, for any fixed input $k' = k_1 \mathinner{..} k_{i-1}k_{i+1} \mathinner{..} k_n$ to Eval we are sampling independent, uniformly generated elements to be used as the PRF keys. Finally endowed with uniform keys, in G_9 the PRF output is replaced with uniform values. If no abort condition is triggered, then the output distributions of G_9 and PR_i^1 are identical.

The complete proof can be found in the full version of the paper [13].

4.2 Split-Key PRFs in the Random Oracle Model

Next, we consider constructions of skPRFs where the key-mixing step employs standard model primitives. However, to achieve security we idealize the PRF that is employed afterwards. Here we identify a sufficient condition on the key-mixing function such that the overall construction achieves split-key pseudorandomness. We begin by giving the aforementioned property for the key-mixing function.

Almost uniformity of a key-mixing function. For all $i \in [1 \mathinner{..} n]$ let $\mathcal{K}_i$ be a finite key space and $\mathcal{K}$ any key space. Consider a function

$$g\colon \mathcal{K}_1 \times \ldots \times \mathcal{K}_n \to \mathcal{K} \ .$$

We say that g is *ϵ-almost uniform w.r.t. the ith key* if for all $k \in \mathcal{K}$ and all $k_j \in \mathcal{K}_j$ for $j \in [1 \mathinner{..} n] \setminus \{i\}$ we have:

$$\Pr_{k_i \leftarrow_\$ \mathcal{K}_i}[g(k_1 \mathinner{..} k_n) = k] \leq \epsilon \ .$$

We say that g is *ϵ-almost uniform* if it is *ϵ-almost uniform w.r.t. the ith key* for all $i \in [1 \mathinner{..} n]$.

We give three standard model instantiations of key-mixing functions that enjoy almost uniformity.

Example 1. Let $\mathcal{K}_1 = \ldots = \mathcal{K}_n = \mathcal{K} = \{0,1\}^k$ for some $k \in \mathbb{N}$ and define

$$g_\oplus(k_1 \mathinner{..} k_n) \coloneqq \bigoplus_{j=1}^{n} k_j \ .$$

Then $g_\oplus$ is $1/\left|\mathcal{K}\right|$-almost uniform.

The proof follows from observing that for any $i \in [1 \mathinner{..} n]$ and any fixed $k_1 \mathinner{..} k_{i-1}k_{i+1} \mathinner{..} k_n$, the function $g_\oplus(k_1 \mathinner{..} k_{i-1} \cdot k_{i+1} \mathinner{..} k_n)$ is a permutation.

Example 2. Let $\mathcal{K}, \mathcal{H}$ be finite and $\pi\colon \mathcal{K} \times \mathcal{H} \to \mathcal{H}$ such that for all $k \in \mathcal{K}$ we have that $\pi(k, \cdot)$ is a permutation on $\mathcal{H}$. Let

$$g(k_1 \mathinner{..} k_n) \coloneqq \pi(k_n, \dots \pi(k_1, 0) \dots) \enspace ,$$

for some $0 \in \mathcal{K}$.

If for all $k \in \mathcal{K}$, $\pi(k, \cdot)$ is a pseudorandom permutation (i.e., π is a blockcipher) then for all i and all $k_1 \mathinner{..} k_{i-1} k_{i+1} \mathinner{..} k_n$ there exists an adversary $\mathcal{A}$ against the pseudorandomness of π such that g is $\mathrm{Adv}_\pi^{\mathrm{prp}}(\mathcal{A}) + 1/\,|\mathcal{K}|$-almost uniform. Here $\mathrm{Adv}_\pi^{\mathrm{prp}}(\mathcal{A})$ is the advantage of $\mathcal{A}$ in distinguishing π under a uniform key from a uniform permutation.

We sketch a proof of Example 2. First, observe that, since k_j for all $j \neq i$ is known by $\mathcal{A}$, all permutations $\pi(k_j, \cdot)$ can be disregarded. Secondly, we replace the permutation $\pi(k_i, \cdot)$ with a uniform permutation, losing the term $\mathrm{Adv}_\pi^{\mathrm{prp}}(\mathcal{A})$. The claim follows.

Example 3. Let $\mathcal{K}_1, \dots, \mathcal{K}_n$, $\mathcal{K}$ be finite. Let

$$g(k_1 \mathinner{..} k_n) \coloneqq k_1 \,\|\, \dots \,\|\, k_n \enspace ,$$

then g is $1/\,|\mathcal{K}|$-almost uniform.

The proof uses the same argument as in Example 1.

We now show that we can generically construct a pseudorandom skPRF from any almost-uniform key-mixing function in the random oracle model.

Lemma 6. *Let $g\colon \mathcal{K}_* \to \mathcal{K}'$ be a function. Let $H\colon \mathcal{K}' \times \mathcal{X} \to \mathcal{Y}$ be a (hash) function. Let*

$$H \diamond g\colon \mathcal{K}_* \times \mathcal{X} \to \mathcal{Y}, \;\; (H \diamond g)(k_1, \dots, k_n, x) \coloneqq H(g(k_1 \mathinner{..} k_n), x) \enspace .$$

If H is modeled as a random oracle then for any adversary $\mathcal{A}$ such that g is ϵ-almost uniform and $\mathcal{A}$ makes at most q_H H *queries and q_e* Eval *queries and all i we have*

$$\mathrm{Adv}_i^{\mathrm{pr}}(\mathcal{A}) \leq q_H \cdot \epsilon \enspace .$$

Proof sketch. Note that any adversary against the pseudorandomness of $H \diamond g$ is given access to Eval and H, the latter implementing a random oracle. Now, intuitively, $\mathcal{A}$ is unlikely to predict the output of the g invocation within an Eval query as g is almost uniform. Hence, $\mathcal{A}$ will not query H on the same input as done within Eval. Thus, even in the real game, the output of Eval is likely to be uniform.

We give a refined analysis next.

Proof (Lemma 6). We bound the distance between the probabilities of $\mathcal{A}$ outputting 1 in game PR_i^0 and PR_i^1. The PR_i^b game is given in Figure 12. For game PR_i^b we performed merely syntactical changes: $\mathcal{A}$ is given access to H via oracle H. Two sets S_E, S_H are initialized as empty and updated in lines 01, 11, 17 and used to define an event in line 05.

```
**Games** $\mathrm{PR}_i^b$
00 $X \leftarrow \emptyset$
01 $S_E, S_{\mathrm{H}} \leftarrow \emptyset$
02 $k_i \leftarrow_{\$} \mathcal{K}_i$
03 $b' \leftarrow_{\$} \mathcal{A}^{\mathrm{Eval},\mathrm{H}}$
04 If $S_{\mathrm{H}} \cap S_E \neq \emptyset$:
05     bad $\leftarrow$ true
06 Stop with $b'$

**Oracle** $\mathrm{Eval}(k', x)$
07 If $x \in X$: Abort
08 $X \leftarrow X \cup \{x\}$
09 $k_1 \mathinner{..} k_{i-1} k_{i+1} \mathinner{..} k_n \leftarrow k'$
10 $k'' \leftarrow g(k_1 \mathinner{..} k_i \mathinner{..} k_n)$
11 $S_E \leftarrow S_E \cup \{(k'', x)\}$
12 If $H[k'', x] = \perp$:
13     $H[k'', x] \leftarrow_{\$} \mathcal{K}'$
14 $y \leftarrow H[k'', x]$
15 $y^0 \leftarrow y$; $y^1 \leftarrow_{\$} \mathcal{Y}$
16 Return $y^b$

**Oracle** $\mathrm{H}(k'', x)$
17 $S_{\mathrm{H}} \leftarrow S_{\mathrm{H}} \cup \{(k'', x)\}$
18 If $H[k'', x] = \perp$:
19     $H[k'', x] \leftarrow_{\$} \mathcal{K}'$
20 Return $H[k'', x]$
```

Fig. 12. Game PR_i^b for $i \in [1 \mathinner{..} n]$ instantiated with $H \diamond g$.

Observe that for all i the PR_i^0 and PR_i^1 games are identical if bad does not happen: As $S_H \cap S_E$ remains empty, adversary $\mathcal{A}$ did not query H on an input that H was evaluated on during an Eval query (see line 14). Hence, $y \leftarrow \mathrm{H}(k'', x)$ is uniform and thus, $y^0 \leftarrow y$ and $y^1 \leftarrow \mathcal{Y}$ are identically distributed.

We bound Pr[bad] in PR_i^1. To this end, let (k''_j, x_j) for $j \in [1 \mathinner{..} q_H]$ denote the H queries made by $\mathcal{A}$. We have

$$\Pr[\text{bad}] = \Pr[S_{\mathrm{H}} \cap S_E \neq \emptyset] \leq \sum_{j=1}^{q_H} \Pr[(k''_j, x_j) \in S_E] \ .$$

Recall from line 07 that for every $x \in \mathcal{X}$ there is at most one query $\mathrm{Eval}(\cdot, x)$ by $\mathcal{A}$. Hence, for each (k''_j, x_j) in S_H there is at most one element of the form $(\cdot, x_j)$ in S_E. Assume it exists[7] and let k''_{x_j} be such that $(k''_{x_j}, x_j) \in S_E$ denotes that element. Then

$$\begin{aligned}\sum_{j=1}^{q_H} \Pr[(k''_j, x_j) \in S_E] &\leq \sum_{j=1}^{q_H} \Pr[k''_j = k''_{x_j}] \\ &= \sum_{j=1}^{q_H} \Pr_{k_{i,x_j} \leftarrow \mathcal{K}_i}[k''_j = g(k'_1 \mathinner{..} k_{i-1} k_{i,x_j} k_{i+1} \mathinner{..} k'_n)]\end{aligned}$$

for $k'_1, \ldots, k'_{i-1}, k'_{i+1}, \ldots, k'_n$ chosen by $\mathcal{A}$ and uniform k_{i,x_j} such that it satisfies $g(k'_1 \mathinner{..} k_{i-1} k_{i,x_j} k_{i+1} \mathinner{..} k'_n) = k''_{x_j}$. Eventually, we can employ the ϵ-almost uniformity of g to conclude that

$$\sum_{j=1}^{q_H} \Pr_{k_{i,x_j} \leftarrow \mathcal{K}_i}[k''_j = g(k'_1 \mathinner{..} k_{i-1} k_{i,x_j} k_{i+1} \mathinner{..} k'_n)] \leq \sum_{j=1}^{q_H} \epsilon \leq q_H \cdot \epsilon \ .$$

□

Next, we show that, generally, the construction from Lemma 6 does not yield a split-key pseudorandom function in the standard model.

[7] If such an element does not exist the following bounds would only become tighter.

Lemma 7. *Let g be with syntax as in Lemma 6 and let F be with syntax as H in Lemma 6. There exists an instantiation of g and F such that g is almost uniform and F is pseudorandom but*

$$F \diamond g \colon \mathcal{K}_* \times \mathcal{X} \to \mathcal{Y} \ , \quad (F \diamond g)(k_1, \ldots, k_n, x) \coloneqq F(g(k_1 \mathinner{..} k_n), x)$$

is not a pseudorandom skPRF.

Proof. We saw in Example 1 that $g_\oplus$ is almost uniform. Further, we saw in Lemma 3 that, when using $F \diamond g_\oplus$ as a core function, there exists a pseudorandom function F such that the combined KEM is not CCA secure. If $F \diamond g_\oplus$ (with such F) were split-key pseudorandom, then this would contradict Theorem 1. □

5 A KEM Combiner in the Standard Model

Our approach was hitherto to mix the keys $k_1, \ldots, k_n$ to obtain a key for a PRF, which was then evaluated on the ciphertext vector. The drawback of this is that to show security we had to turn to idealized primitives. In the following we embark on a different approach, with the goal to obtain a standard model construction.

5.1 The PRF-Then-XOR Split-Key PRF

Here we abstain from mixing the keys together, but use each key k_i in a PRF evaluation. The security of the model is offset by its price in terms of efficiency: When employed in a parallel combiner, the skPRF requires n PRF calls, whereas for our constructions secure in idealized models in Section 4.2 a single call to a PRF suffices. We give our construction next.

As before we want to allow possibly different session-key spaces of the ingredient KEMs. Thus, as the keys k_i in Construction 2 come from an encapsulation of K_i, we allow the construction to use distinct PRFs. Yet, one may choose $F_i = F_j$ for all i, j, if supported by the ingredient KEM's syntax.

Construction 2. *For all $i \in [1 \mathinner{..} n]$ let $F_i \colon \mathcal{K}_i \times \mathcal{X} \to \mathcal{Y}$ be a function and let $\mathcal{K} = \mathcal{K}_1 \times \ldots \times \mathcal{K}_n$. We define the PRF-then-XOR composition of $F_1, \ldots, F_n$:*

$$[F_1 \mathinner{..} F_n] \colon \mathcal{K} \times \mathcal{X} \to \mathcal{Y} \ , \quad [F_1 \mathinner{..} F_n](k_1, \ldots, k_n, x) \coloneqq \bigoplus_{i=1}^{n} F_i(k_i, x) \ .$$

Lemma 8. *For all $i \in [1 \mathinner{..} n]$ let F_i be as in Construction 2. If all F_i are pseudorandom then $[F_1 \mathinner{..} F_n]$ is split-key pseudorandom.*

More precisely, for all $n, F_1, \ldots, F_n$, for all indices i and all adversaries $\mathcal{A}$ there exist an adversary $\mathcal{B}$ such that

$$\mathrm{Adv}^{\mathrm{pr}}_{[F_1 \mathinner{..} F_n], i}(\mathcal{A}) \leq \mathrm{Adv}^{\mathrm{pr}}_{F_i}(\mathcal{B}) \ .$$

Suppose that $\mathcal{A}$ poses at most q queries to its evaluation oracle. Then adversary $\mathcal{B}$ poses at most q queries to its own encapsulation oracle. The running times of $\mathcal{B}$ is roughly the same as of $\mathcal{A}$.

Proof. We fix an index $i \in [1 \mathinner{..} n]$ and we build an adversary $\mathcal{B}$ against the PRF F_i from an adversary $\mathcal{A}$ against the skPRF $[F_1 \mathinner{..} F_n]$.

Adversary $\mathcal{B}$ works as follows. It starts by running adversary $\mathcal{A}$. Each time that $\mathcal{A}$ queries the oracle Eval on input (k', x) it queries its own evaluation oracle on input x, obtaining the output $y \in \mathcal{Y}$. Then it computes the key $k := y \oplus \bigoplus_{j \neq i} F_j(k_j, x)$, and returns the key to $\mathcal{A}$. Finally, $\mathcal{B}$ returns the output of $\mathcal{A}$.

We observe that if $\mathcal{B}$ is playing against game PR^0 then it receives a real evaluation of F_i from the oracle Eval. Hence $\mathcal{B}$ returns to $\mathcal{A}$ a real key and $\mathcal{A}$ is playing against game PR_i^0. If $\mathcal{B}$ is playing against game PR^1 instead, then $\mathcal{B}$ receives independent, uniformly distributed values from the oracle Eval (note that, by the restrictions of game PR_i^1, adversary $\mathcal{A}$ queries its oracle on distinct input each time). If we add any constant value to $y \leftarrow_{\$} \mathcal{Y}$ the result remains uniformly distributed. Hence, on each query to Eval adversary $\mathcal{B}$ returns to $\mathcal{A}$ independent uniformly distributed keys and $\mathcal{A}$ is playing against game PR_i^1. $\square$

Note that Lemma 8 gives raise to a standard model KEM combiner that requires n PRF invocations, each processing the concatenation of n encapsulations $c = c_1 \parallel \ldots \parallel c_n$. For a slightly more efficient combiner where each of the n PRF invocations is evaluated on the concatenation of $n-1$ encapsulations see the full version of this paper [13].

Acknowledgments

We are grateful to the anonymous PKC reviewers for their valuable comments.

Federico Giacon was supported by ERC Project ERCC (FP7/615074). Felix Heuer was supported by Mercator Research Center Ruhr project "LPN-Krypt: Das LPN-Problem in der Kryptographie". Bertram Poettering conducted part of this work at Ruhr University Bochum, supported by ERC Project ERCC (FP7/615074).

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