# Low Noise LPN: KDM Secure Public Key Encryption and Sample Amplification

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**Abstract.** Cryptographic schemes based on the Learning Parity with Noise (LPN) problem have several very desirable aspects: Low computational overhead, simple implementation and conjectured post-quantum hardness. Choosing the LPN noise parameter sufficiently low allows for public key cryptography. In this work, we construct the first standard model public key encryption scheme with key dependent message security based solely on the low noise LPN problem. Additionally, we establish a new connection between LPN with a bounded number of samples and LPN with an unbounded number of samples. In essence, we show that if LPN with a small error and a small number of samples is hard, then LPN with a slightly larger error and an unbounded number of samples is also hard. The key technical ingredient to establish both results is a variant of the LPN problem called the extended LPN problem.

**Keywords:** Low Noise LPN, Key Dependent Message Security, LPN Hardness Reduction

## 1 Introduction

*The LPN Problem* The learning parity with noise (LPN) problem asks to find a secret binary vector $\mathbf{s} \in \mathbb{F}_2^n$ given noisy linear samples of the form $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e) \in \mathbb{F}_2^n \times \mathbb{F}_2$ where $\mathbf{a}$ is chosen uniformly at random and $e$ is an additive noise term that occurs with probability $\rho$. Due to its simplicity and binary arithmetic, the LPN problem has become a central hub in secret key cryptography [13,28,30,27,33]. These applications use the *high noise* LPN problem where the noise rate $\rho < 1/2$ is a constant. In the *low noise* LPN problem, the noise rate $\rho$ tends asymptotically to 0. Alekhnovich [6] provided a construction of a public key encryption scheme based on LPN for noise rates $\rho = O(1/\sqrt{n})$. Recently, more complex cryptographic primitives have been constructed from low noise

LPN such as chosen ciphertext secure public key encryption [22,31] and composable oblivious transfer [18]. In the original formulation of the LPN problem, the search algorithm/adversary may demand an unbounded number of samples whereas the bounded samples version (e.g. used in [6,22,31]) only provides an a priori bounded number of samples to the search algorithm. So far, it was unknown whether the hardness of LPN with a bounded number of samples implies the hardness of LPN with an unbounded number of samples, even if a modest increase in the noise rate is tolerated.

*Key Dependent Message Security* A public key encryption scheme is called key dependent message (KDM) secure, if encryptions of the secret key, or more generally encryptions of functions of several secret keys are indistinguishable of encryptions of (say) the all-zero message. We will exclusively consider KDM-CPA security in this work, i.e. KDM adversaries do not have access to a decryption oracle. While for most natural cryptographic tasks standard notions of security are sufficient, the notion of KDM security is relevant for contexts such as computational soundness [12,2] or when hard-disks are encrypted that store the corresponding secret key (as mentioned in [15]). It has been shown that standard IND-CPA (or even IND-CCA) security does not imply KDM security [1,16], i.e. there exist public key encryption schemes with IND-CPA security relative to some standard assumption which are provably not KDM secure. Standard model KDM secure public key public key cryptosystems were constructed from a variety of assumptions, starting with the construction of Boneh et al [15]. Applebaum et al. [10] provided both a circular secure public key encryption scheme from the LWE assumption and a circular secure *private key* encryption scheme from the (high noise) LPN problem. The latter scheme was later shown to fulfill the stronger notion of related-key KDM security by Applebaum [9]. In [8], Applebaum provided a construction of a KDM secure PKE for arbitrary (bounded size) circuits from any KDM secure PKE for affine functions. Constructing a KDM secure public key encryption scheme from low noise LPN has remained an open problem so far.

## 1.1 Extended LPN

The central tool we use in our constructions is a version of the LPN problem called *extended decisional LPN problem*, or eDLPN in short. The eDLPN problem can be seen as a special case for $q = 2$ of the extended LWE problem introduced O'Neill, Peikert and Waters [35] and proven hard under standard LWE by Alperin-Sheriff and Peikert [7]. The binary version we use in this work was first discussed by Kiltz, Masny and Pietrzak [31].

In the eDLPN problem, the adversary's goal is to distinguish $(\mathbf{A}, \mathbf{RA}, \mathbf{e}, \mathbf{Re})$ from $(\mathbf{A}, \mathbf{U}, \mathbf{e}, \mathbf{Re})$, where $\mathbf{A}$ is a randomly chosen matrix, $\mathbf{R}$ is a randomly chosen low weight matrix, $\mathbf{U}$ is a randomly chosen matrix and $\mathbf{e}$ follows some distribution $\boldsymbol{\chi}$. This is similar to the dual formulation of the decisional LPN problem, where the adversary has to distinguish $(\mathbf{A}, \mathbf{RA})$ from $(\mathbf{A}, \mathbf{U})$. However, in the extended decisional LPN problem, the adversary obtains an extra advice

**Re** about a secret matrix $\mathbf{R}$, where the vector $\mathbf{e}$ can have any distribution. Kiltz, Masny and Pietrzak [31] observed that in the LPN case, this advice can be extremely useful to enable reductions to simulate faithfully. In particular, the eDLPN problem can effectively be used as a computational substitute for the (generalized) leftover hash lemma [29,19] or gaussian regularity lemmata for lattices [26].

In the full version [21], we provide a generalization of the extended LPN problem we call leaky LPN ($\ell$-LPN), which may be of independent interest. In the $\ell$-LPN problem, the advice given to the adversary can be described by an arbitrary adversarially chosen leakage function $\gamma$ from a family $\mathcal{L}$ and is not limited to linear functions as in the extended LPN problem. Clearly, the hardness of the extended LPN problem follows immediately from the hardness of the leaky LPN problem when instantiating the leakage functions with linear functions. If the functions in $\mathcal{L}$ output short strings, say strings of at most logarithmic length, then the hardness of the $\ell$-LPN *search* problem follows immediately from the standard LPN problem, since all possible leakage values can be efficiently enumerated (or guessed). The situation is slightly different for decisional problems. In general, decisional problems become easy if even a single bit of arbitrary leakage is allowed. However, we only allow the leakage to depend on $\mathbf{R}$ and in particular not on $\mathbf{A}$. We show that a sample preserving search to decision reduction of Applebaum et al. [11] is in fact *leakage preserving*. We can thus base the hardness of the decisional problem $\ell$-DLPN on $\ell$-LPN, and therefore on LPN given that the functions in $\mathcal{L}$ only provide short advice.

## 1.2 KDM Secure Public Key Encryption

We will now provide an overview of our construction of a KDM secure public key encryption scheme from LPN. The construction is inspired by the public key encryption scheme of Applebaum et al. [10], which however lives in the LWE realm. The basic idea, as in [10], is to make encryptions of the secret key syntactically similar to the public key. More specifically, public keys in our scheme will be of the form $(\mathbf{A}, \mathbf{y} = \mathbf{As} + \mathbf{e})$ where $\mathbf{s}$ is the secret key. It follows immediately from the decisional LPN problem that the public key is pseudorandom. Encryption takes a message $\mathfrak{m}$ and computes

$$\mathbf{C}_1 = \mathbf{RA}$$
$$\mathbf{c}_2 = \mathbf{Ry} + \mathbf{G}\mathfrak{m},$$

where the matrix $\mathbf{R}$ is chosen from a low weight distribution and $\mathbf{G}$ is the generator matrix of a good, efficiently decodable binary linear code. We remark that while this scheme bears strong resemblances with (and is inspired by) the LWE based scheme of [10], it is rather incomparable to the (high noise) LPN based private key encryption schemes of [10,9] or previous low-noise LPN public key encryption schemes [6,22,31]. Notice that standard IND-CPA security of this scheme follows directly from the fact that $\mathbf{y}$ is pseudorandom and thus also

$(\mathbf{RA}, \mathbf{Ry})$ is pseudorandom given the public key $(\mathbf{A}, \mathbf{y})$, by using the dual formulation of the decisional LPN problem (i.e. $(\mathbf{A}', \mathbf{RA}') \approx_c (\mathbf{A}', \mathbf{U})$). To decrypt a ciphertext $\mathfrak{c} = (\mathbf{C}_1, \mathbf{c}_2)$, we basically compute

$$\mathbf{z} = \mathbf{c}_2 - \mathbf{C}_1 \mathbf{s}$$

and recover $\mathfrak{m}$ from $\mathbf{z}$ by using the efficient decoding algorithm for the code generated by $\mathbf{G}$. Correctness of the scheme follows from the fact that

$$\begin{aligned}
\mathbf{z} &= \mathbf{c}_2 - \mathbf{C}_1 \mathbf{s} \\
&= \mathbf{Ry} + \mathbf{Gm} - \mathbf{RAs} \\
&= \mathbf{R}(\mathbf{As} + \mathbf{e}) + \mathbf{Gm} - \mathbf{RAs} \\
&= \mathbf{Gm} + \mathbf{Re}.
\end{aligned}$$

Since we have chosen $\mathbf{R}$ and $\mathbf{e}$ from low noise distributions, the term $\mathbf{Re}$ has low weight with high probability. Thus it follows that a decoder of the code generated by $\mathbf{G}$ will be able to recover $\mathfrak{m}$ from $\mathbf{z}$. We will briefly sketch how to establish 1-circular security of this scheme, where the adversary gets a single encryption of the secret key (or an encryption of 0). For the full proof of KDM security for affine functions, refer to Section 3. An encryption of the secret key has the form $(\mathbf{RA}, \mathbf{Ry} + \mathbf{Gs})$. Figure 1 provides the game transform for this security reduction.

| | | Game public key | challenge ciphertext | remark |
|---|---|---|---|---|
| 1. | Real | $(\mathbf{A}, \mathbf{y} = \mathbf{As} + \mathbf{e})$ | $(\mathbf{RA}, \mathbf{Ry} + \mathbf{Gs})$ | |
| 2. | Real | $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ | $(\mathbf{RA}, \mathbf{R}(\mathbf{As} + \mathbf{e}) + \mathbf{Gs})$ | identical |
| 3. | Real | $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ | $(\mathbf{RA}, (\mathbf{RA} + \mathbf{G})\mathbf{s} + \mathbf{Re})$ | identical |
| 4. | $\mathbf{H}_1$ | $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ | $(\mathbf{U}, (\mathbf{U} + \mathbf{G})\mathbf{s} + \mathbf{Re})$ | eDLPN |
| 5. | $\mathbf{H}_1$ | $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ | $(\mathbf{U}' - \mathbf{G}, \mathbf{U}'\mathbf{s} + \mathbf{Re})$ | identical |
| 6. | $\mathbf{H}_2$ | $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ | $(\mathbf{RA} - \mathbf{G}, \mathbf{RAs} + \mathbf{Re})$ | eDLPN |
| 7. | $\mathbf{H}_2$ | $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ | $(\mathbf{RA} - \mathbf{G}, \mathbf{R}(\mathbf{As} + \mathbf{e}))$ | identical |
| 8. | $\mathbf{H}_3$ | $(\mathbf{A}, \mathbf{u})$ | $(\mathbf{RA} - \mathbf{G}, \mathbf{Ru})$ | DLPN |
| 9. | $\mathbf{H}_3$ | $(\mathbf{A}, \mathbf{u})$ | $(\mathbf{U} - \mathbf{G}, \mathbf{u}')$ | DDLPN |
| 10. | $\mathbf{H}_3$ | $(\mathbf{A}, \mathbf{u})$ | $(\mathbf{U}, \mathbf{u}')$ | identical |

**Fig. 1.** The Game Transform for KDM-CPA security

The first three steps shown in Figure 1 do not change the real experiment but basically rewrite the challenge ciphertext. From step 3 to step 4 we replace the matrix $\mathbf{RA}$ by a uniformly random matrix $\mathbf{U}$. Since we also need the additional term $\mathbf{Re}$ to provide the correct distribution to the adversary, we will use the extended decisional LPN problem to show that these two experiments are computationally indistinguishable. In particular, we use $\mathbf{A}$ and $\mathbf{e}$ provided by the eDLPN problem to construct a public key, while we use $\mathbf{RA}$ and the advice $\mathbf{Re}$ to construct the encryption of the secret key. Then, we replace $\mathbf{RA}$ by a random matrix $\mathbf{U}$, which yields an indistinguishable experiment by the hardness

of eDLPN. Step 4 to 5 is another bridging step which does not change the experiment. Since $\mathbf{U}$ is distributed uniformly random, so is the matrix $\mathbf{U}' = \mathbf{U} + \mathbf{G}$. Thus, instead of choosing $\mathbf{U}$ uniformly at random we can choose $\mathbf{U}'$ uniformly at random and set $\mathbf{U} = \mathbf{U}' - \mathbf{G}$. From step 5 to step 6 we replace the matrix $\mathbf{U}'$ by $\mathbf{RA}$. Again, we have to use the extended decisional LPN problem as we need the extra advice $\mathbf{Re}$. It now becomes clear that we have used steps 3 to 6 to *pull* the matrix $\mathbf{G}$ from the second component of the challenge ciphertext to its first component, i.e. we have transformed $(\mathbf{RA}, (\mathbf{RA} + \mathbf{G})\mathbf{s} + \mathbf{Re})$ into $(\mathbf{RA} - \mathbf{G}, \mathbf{RAs} + \mathbf{Re})$. Step 6 to step 7 is another basic bridging step. From step 7 to step 8 we replace the second component $\mathbf{As} + \mathbf{e}$ of the public key by a randomly chosen $\mathbf{u}$, indistinguishability follows from the standard decisional LPN problem. From step 8 to step 9 we replace $(\mathbf{A}, \mathbf{RA}, \mathbf{u}, \mathbf{Ru})$ by $(\mathbf{A}, \mathbf{U}, \mathbf{u}, \mathbf{u}')$ for uniformly random $\mathbf{U}$ and $\mathbf{u}'$, indistinguishability follows from the dual formulation of the decisional LPN problem. Finally, from step 9 to step 10, we replace $\mathbf{U} - \mathbf{G}$ by $\mathbf{U}$. We can do this since the uniform distribution $\mathbf{U}$ is invariant under an additive shift by a constant matrix $\mathbf{G}$. Thus, in the last experiment the challenge ciphertext is just uniformly random, which concludes this outline.

### 1.3 Unbounded Samples LPN from Bounded Samples LPN

In the following we will distinguish between bounded and unbounded samples LPN. We will denote search LPN with a secret of length $n$, $m$ samples and noise rate $\rho$ by $\mathsf{LPN}(n, m, \rho)$ and decisional LPN with a secret of length $n$, unbounded samples and noise rate $\rho'$ by $\mathsf{DLPN}(n, \rho')$. Our second contribution is a hardness reduction which bases the hardness of $\mathsf{DLPN}(n, \rho')$ on $\mathsf{LPN}(n, 2n, \rho)$. More specifically, we show that if $\mathsf{LPN}(n, 2n, \rho)$ is hard, then $\mathsf{DLPN}(n, \rho')$ is also hard, where

$$\rho' = \frac{1}{2} - \frac{1}{2}(1 - 2\rho)^{\lfloor \rho 2n \rfloor} \leq 2\rho^2 n.$$

For the Learning With Errors (LWE) problem, there exists a statistical *random self reduction* [26,10]. The idea of this reduction is to use $m \approx n \log(q)$ *seed samples* to generate arbitrarily many fresh samples. The noise rate in the new samples increases only slightly. Specifically, if $(\mathbf{A}, \mathbf{y} = \mathbf{As} + \mathbf{z})$ is such a given set of *seed samples*, then one can generate new samples by drawing $\mathbf{e} \in \mathbb{Z}_q^m$ from a discrete gaussian [5,34] and setting $\mathbf{a}' = \mathbf{A}^\top \mathbf{e}$ and $y' = \mathbf{e}^\top \mathbf{y}$. Now it holds

$$y' = \mathbf{e}^\top \mathbf{y} = \mathbf{e}^\top \mathbf{As} + \mathbf{e}^\top \mathbf{z} = \mathbf{a}'^\top \mathbf{s} + \langle \mathbf{e}, \mathbf{z} \rangle.$$

The pair $(\mathbf{a}', y' + e')$, where $e'$ is a gaussian smoothing term, is a proper LWE sample, as $\mathbf{a}' = \mathbf{e}^\top \mathbf{A}$ can be shown to be statistically close to uniform *and* $\langle \mathbf{e}, \mathbf{z} \rangle$ follows an independent discrete gaussian distribution even conditioned on $\mathbf{a}' = \mathbf{e}^\top \mathbf{A}$ [1].

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[1]This can be established via a Lemma due to Regev [37] or its refinement due to Peikert [36], which show that the distribution of $\mathbf{e}$ remains discrete gaussian even conditioned on $\mathbf{e}^\top \mathbf{A}$, though the variance of the distribution decreases.

Such an approach, however, cannot be directly transferred to the LPN setting. For the vector $\mathbf{a}' = \mathbf{A}^\top \mathbf{e}$ to be statistically close to uniform, $\mathbf{e}$ must have min-entropy $\approx n$, and thus high weight. But this in turn means that $\langle \mathbf{e}, \mathbf{z} \rangle$ will only have a negligibly small bias. We remark that such a high noise sample amplification was used Lyubashevsky [32] to cryptanalize LPN in sub-exponential time, but this technique does not seem to be applicable in the context of an *efficient* (i.e. PPT) hardness reduction, especially when the number of samples is at most polynomial.

Therefore, in our reduction we will replace the statistical tools in the above reduction by a computational technique based on the eDLPN problem. Again, we start with a given amount of $m = 2n$ seed samples and generate new samples from these. While we cannot hope that the samples we generate in this way have the proper distribution (in the statistical sense), we will be able to show that the distribution generated in this way is computationally indistinguishable from the real LPN distribution. More specifically, let $(\mathbf{A}, \mathbf{y})$ be the LPN seed samples. We will compute new samples by choosing a random low weight $\mathbf{r}$ and setting $\mathbf{a} = \mathbf{A}^\top \mathbf{r}$ and $y' = \mathbf{r}^\top \mathbf{y} = \langle \mathbf{r}, \mathbf{y} \rangle$. Now, assume first that $\mathbf{y} = \mathbf{As} + \mathbf{e}$. Then it holds that

$$y' = \mathbf{r}^\top \mathbf{As} + \langle \mathbf{r}, \mathbf{e} \rangle = \langle \mathbf{a}, \mathbf{s} \rangle + \langle \mathbf{r}, \mathbf{e} \rangle.$$

While $(\mathbf{a}, y')$ syntactically looks like an LPN sample, it is statistically far away from a correctly distributed sample. There are two issues. First, $\mathbf{a} = \mathbf{r}^\top \mathbf{A}$ is not distributed uniformly. Second, the noise term $\langle \mathbf{r}, \mathbf{e} \rangle$ is correlated with $\mathbf{a}$. The first issue *alone* could be resolved by assuming the hardness of the DLPN. To deal with both issues simultaneously, we will resort to the eDLPN problem, which allows us to present a noise term $\langle \mathbf{r}, \mathbf{e} \rangle$ with the right distribution. More specifically, the eDLPN problem allows us to replace $\mathbf{a} = \mathbf{r}^\top \mathbf{A}$ by a uniformly random $\mathbf{a}$ but also provides us with an *advice* $(\mathbf{e}, \langle \mathbf{r}, \mathbf{e} \rangle)$ that allows us to simulate the noise term $\langle \mathbf{r}, \mathbf{e} \rangle$ correctly. On the other hand, if $\mathbf{y}$ was chosen uniformly at random, then the pseudorandomness of $(\mathbf{a}, y') = (\mathbf{r}^\top \mathbf{A}, \mathbf{r}^\top \mathbf{y})$ follows easily from the dual formulation of the LPN problem DLPN. Since we can base the hardness of all auxiliary problems on $\mathsf{LPN}(n, 2n, \rho)$, it follows that $\mathsf{DLPN}(n, \rho')$ is at least as hard as $\mathsf{LPN}(n, 2n, \rho)$. This concludes this outline.

## 2 Preliminaries

In the following, let $\lambda$ always denote the security parameter. We call a machine PPT if it runs in probabilistic (expected) polynomial time. For a search problem P and an adversary/search algorithm $\mathcal{A}$ let $\mathsf{Adv}_\mathsf{P}(\mathcal{A})$ denote the probability of $\mathcal{A}$ finding a solution of a random instance of P. For a decisional problem D which consists in distinguishing two distributions $X$ and $Y$ and a distinguishing algorithm $\mathcal{D}$ define $\mathsf{Adv}_\mathsf{D}(\mathcal{D}) = |\Pr[\mathcal{D}(X) = 1] - \Pr[\mathcal{D}(Y) = 1]|$. When we don't write it explicitly, we will implicitly assume that search algorithms and distinguishers get $1^\lambda$ as an additional input. We will denote the Hamming weight of a vector $\mathbf{x} \in \mathbb{F}_2^n$ by $\|\mathbf{x}\|_0 = |\{i : x_i \neq 0\}|$. For a matrix $\mathbf{M} \in \mathbb{F}_2^{m \times n}$, we define

the Hamming weight of $\mathbf{M}$ by $\|\mathbf{M}\|_0 = \max_i \|\mathbf{m}_i\|_0$ where the $\mathbf{m}_i$ are the column vectors of $\mathbf{M}$. It follows easily for all $\mathbf{M} \in \mathbb{F}_2^{m \times n}$ and $\mathbf{x} \in \mathbb{F}_2^n$ that $\|\mathbf{Mx}\|_0 \leq \|\mathbf{M}\|_0 \cdot \|\mathbf{x}\|_0$. We need asymptotically good, efficiently decodable binary linear codes for the construction of our KDM secure public key encryption scheme. A binary linear $[k, n]$ code $\mathsf{C}$ is a $n$ dimensional subspace of $\mathbb{F}_2^k$. We call $\mathbf{G} \in \mathbb{F}_2^{k \times n}$ a generator matrix of $\mathsf{C}$ if every $\mathbf{c} \in \mathsf{C}$ can be written as $\mathbf{c} = \mathbf{Gx}$ for some $\mathbf{x} \in \mathbb{F}_2^n$. We assume codes $\mathsf{C}$ come with efficient encoding and decoding procedures $\mathsf{C.Encode}$ and $\mathsf{C.Decode}$, where $\mathsf{C.Encode}(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x}$ for some generator matrix $\mathbf{G}$ of $\mathsf{C}$. An error correcting code can efficiently correct an $\alpha$ fraction of errors, if for every $\mathbf{e} \in \mathbb{F}_2^k$ with $\|\mathbf{e}\|_0 \leq \alpha k$, it holds that $\mathsf{C.Decode}(\mathsf{C.Encode}(\mathbf{x}) + \mathbf{e}) = \mathbf{x}$. There exists a large corpus of literature of linear codes that can efficiently correct a constant fraction of errors, for instance concatenated codes [25] or expander codes [38,39].

### 2.1 Learning Parity with Noise

We will denote the Bernoulli distribution with parameter $\rho \in [0, 1/2]$ on $\mathbb{F}_2^m$ by $\mathsf{Ber}(m, \rho)$. For an $\mathbf{e} \leftarrow_\$ \mathsf{Ber}(m, \rho)$, each component $e_i$ of $\mathbf{e}$ independently takes the value 1 with probability $\rho$ and 0 with probability $1 - \rho$. We write $\mathsf{Ber}(\rho) := \mathsf{Ber}(1, \rho)$. We will distinguish between LPN with a bounded and an unbounded number of samples.

**Definition 1 (Learning Parity with Noise).** *Let $\boldsymbol{\chi}$ be an error distribution on $\mathbb{F}_2^m$ and $\rho = \rho(\lambda) \in [0, 1/2]$. Let $\mathbf{A} \leftarrow_\$ \mathbb{F}_2^{m \times n}$ be chosen uniformly at random, let $\mathbf{s} \leftarrow_\$ \mathbb{F}_2^n$ be chosen uniformly at random and let $\mathbf{e} \leftarrow_\$ \boldsymbol{\chi}$.*

1. *In the bounded samples search problem $\mathsf{LPN}(n, m, \boldsymbol{\chi})$, the goal is to find $\mathbf{s}$, given $(\mathbf{A}, \mathbf{As} + \mathbf{e})$.*
2. *In the unbounded samples search problem $\mathsf{LPN}(n, \rho)$, the goal is to find $\mathbf{s}$, given an oracle that outputs an arbitrary number of samples of the form $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e)$, where $\mathbf{a} \leftarrow_\$ \mathbb{F}_2^n$ and $e \leftarrow_\$ \mathsf{Ber}(\rho)$.*
3. *In the bounded samples decisional problem $\mathsf{DLPN}(n, m, \boldsymbol{\chi})$, the goal is to distinguish the distributions $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ and $(\mathbf{A}, \mathbf{u})$, where $\mathbf{u} \leftarrow_\$ \mathbb{F}_2^m$ is chosen uniformly at random.*
4. *In the unbounded samples decisional problem $\mathsf{DLPN}(n, \rho)$, the goal is to distinguish two oracles, namely one that outputs samples of the form $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e)$ (where $\mathbf{a} \leftarrow_\$ \mathbb{F}_2^n$ and $e \leftarrow_\$ \mathsf{Ber}(\rho)$) from one that outputs samples of the form $(\mathbf{a}, u)$ (where $\mathbf{a} \leftarrow_\$ \mathbb{F}_2^n$ and $u \leftarrow_\$ \mathbb{F}_2$).*

For bounded samples LPN with errors $\mathbf{e}$ from the Bernoulli distribution $\mathsf{Ber}(m, \rho)$ we will write $\mathsf{LPN}(n, m, \rho)$ for $\mathsf{LPN}(n, m, \mathsf{Ber}(m, \rho))$ and also $\mathsf{DLPN}(n, m, \rho)$ for $\mathsf{DLPN}(n, m, \mathsf{Ber}(m, \rho))$. By a standard argument, one can show that if $\mathbf{e} \in \mathbb{F}_2^m$ is distributed according to $\mathsf{Ber}(m, \rho)$ and $\mathbf{z} \in \mathbb{F}_2^m$ is an arbitrary vector of weight $\lfloor \rho m \rfloor$, then $\langle \mathbf{z}, \mathbf{e} \rangle$ is distributed according to $\mathsf{Ber}(\rho')$, where $\rho' = \frac{1}{2} - \frac{1}{2}(1 - 2\rho)^{\lfloor \rho m \rfloor} \leq \rho^2 m$. Following Alekhnovich [6], we will choose the noise parameter $\rho$ of the form $O(1/\sqrt{n})$ and $n, m = \Omega(\lambda^2)$ to be able to use low weight vectors as trapdoors and have $2^\lambda$ (conjectured) security for $\mathsf{LPN}(n, m, \rho)$.

A series of works have established relations between search and decisional LPN problems [13,30,11]. The hardness reduction of Applebaum et al. [11] is sample preserving, i.e. it shows that the hardness of $\mathsf{DLPN}(n, m, \boldsymbol{\chi})$ follows directly from the hardness of $\mathsf{LPN}(n, m, \boldsymbol{\chi})$, for any error distribution $\boldsymbol{\chi}$.

**Lemma 1 (Applebaum et al. [11]).** *Let $\boldsymbol{\chi}$ be an error distribution on $\mathbb{F}_2^m$ and assume that $\mathsf{LPN}(n, m, \boldsymbol{\chi})$ is hard. Then $\mathsf{DLPN}(n, m, \boldsymbol{\chi})$ is also hard. More specifically, assume there exists a PPT adversary $\mathcal{A}$ that distinguishes $\mathsf{DLPN}(n, m, \boldsymbol{\chi})$ with advantage $\epsilon$. Then there exists a PPT adversary $\mathcal{A}'$ that breaks $\mathsf{LPN}(n, m, \boldsymbol{\chi})$ with advantage $\epsilon^2/8$.*

Let $\mathsf{S}(m, \rho)$ denote the distribution on $\mathbb{F}_2^m$ which outputs uniformly random vectors in $\mathbb{F}_2^m$ of weight $\lfloor \rho m \rfloor$, i.e. $\mathsf{S}(m, \rho)$ is the uniform distribution on the set $M = \{ \mathbf{x} \in \mathbb{F}_2^m \mid \|\mathbf{x}\|_0 = \lfloor \rho m \rfloor \}$. It is easy to see that if $\mathsf{LPN}(n, m, \rho)$ is hard, then $\mathsf{LPN}(n, m, \mathsf{S}(m, \rho))$ is also hard.

**Corollary 1.** *Let $\mathcal{A}$ be a PPT adversary that breaks $\mathsf{LPN}(n, m, \mathsf{S}(m, \rho))$ with advantage $\epsilon$. Then there exists a PPT adversary $\mathcal{A}'$ that breaks $\mathsf{LPN}(n, m, \rho)$ with advantage $\frac{(1-o(1))\epsilon}{\sqrt{2\pi m \rho(1-\rho)}}$. Moreover, if there exists a PPT distinguisher $\mathcal{D}$ that distinguishes $\mathsf{DLPN}(n, m, \mathsf{S}(m, \rho))$ with advantage $\epsilon$, then there exists a PPT adversary $\mathcal{A}'$ that breaks $\mathsf{LPN}(n, m, \rho)$ with advantage $\frac{(1-o(1))\epsilon^2}{8 \cdot \sqrt{2\pi m \rho(1-\rho)}}$.*

For a proof of Corollary 1, refer to the full version [21]. As a convenient reformulation of the LPN problem, we define the decisional dual LPN problem.

**Definition 2.** *Let $\mathbf{A} \leftarrow_\$ \mathbb{F}_2^{m \times n}$, $\mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$, $\mathbf{a} \leftarrow_\$ \mathbb{F}_2^m$, $\mathbf{U} \leftarrow_\$ \mathbb{F}_2^{k \times n}$ and $\mathbf{u} \leftarrow_\$ \mathbb{F}_2^k$. The goal of the $\mathsf{DDLPN}(n, m, k, \rho)$ problem is to distinguish the distributions $(\mathbf{A}, \mathbf{RA}, \mathbf{a}, \mathbf{Ra})$ and $(\mathbf{A}, \mathbf{U}, \mathbf{a}, \mathbf{u})$.*

The hardness of $\mathsf{DDLPN}(n, m, k, \rho)$ follows from $\mathsf{DLPN}(n, m, \rho)$ (see e.g. [10] or [22]) using the fact that for a randomly chosen matrix $\mathbf{A}$ we can also sample a random $\mathbf{H}$ such that it holds $\mathbf{H} \cdot \mathbf{A} = \mathbf{0}$ and $\mathbf{H}$ is uniformly random (not given $\mathbf{A}$).

**Lemma 2.** *Let $m \geq 2n$. Assume there exists a PPT distinguisher $\mathcal{D}$ that distinguishes the problem $\mathsf{DDLPN}(n, m, k, \rho)$ with advantage $\epsilon$. Then there exists a PPT adversary $\mathcal{A}$ that breaks $\mathsf{LPN}(n, m, \rho)$ with advantage $\frac{\epsilon^2}{8k^2}$.*

Following Kiltz et al. [31] and Alperin-Sheriff and Peikert [7], we provide a definition of the extended LPN problem. We only define the extended LPN problem for Bernoulli error distributions.

**Definition 3.** *Let $\chi$ be any distribution on $\mathbb{F}_2^m$. Let $\mathbf{A} \leftarrow_\$ \mathbb{F}_2^{m \times n}$, $\mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$, $\mathbf{U} \leftarrow_\$ \mathbb{F}_2^{k \times n}$ and $\mathbf{e} \leftarrow_\$ \chi$. The goal of the $\mathsf{eDLPN}(n, m, k, \rho, \chi)$ problem is to distinguish the distributions $(\mathbf{A}, \mathbf{RA}, \mathbf{e}, \mathbf{Re})$ and $(\mathbf{A}, \mathbf{U}, \mathbf{e}, \mathbf{Re})$.*

The hardness of $\mathsf{eDLPN}(n, m, k, \rho, \boldsymbol{\chi})$ can be established from $\mathsf{LPN}(n, m, \rho)$.

**Lemma 3.** *Let $m \geq 2n$. Then for any distribution $\boldsymbol{\chi}$ on $\mathbb{F}_2^m$ and any $k = \mathsf{poly}(\lambda)$ it holds that if there exists a PPT distinguisher $\mathcal{D}$ that distinguishes $\mathsf{eDLPN}(n, m, k, \rho, \boldsymbol{\chi})$ with advantage $\epsilon$, then there exists a PPT adversary $\mathcal{A}$ that breaks $\mathsf{LPN}(n, m, \rho)$ with advantage $\frac{\epsilon^2}{8k^2}$.*

For a proof of Lemma 3 we refer the reader either to [7] or the full version of this paper [21].

### 2.2 Key Dependent Message Secure Public Key Encryption

Syntactically, a public key encryption scheme $\mathsf{PKE}$ consists of three PPT algorithms $\mathsf{PKE.KeyGen}$, $\mathsf{PKE.Enc}$ and $\mathsf{PKE.Dec}$, such that $\mathsf{PKE.KeyGen}$ generates a pair $(pk, sk)$ of public and secret keys, $\mathsf{PKE.Enc}$ takes a public key $pk$ and a plaintext $\mathfrak{m}$ and outputs a ciphertext $\mathfrak{c}$ and $\mathsf{PKE.Dec}$ takes a secret key $sk$ and a ciphertext $\mathfrak{c}$ and outputs a plaintext $\mathfrak{m}$. We say that $\mathsf{PKE}$ is correct, if it holds for all plaintexts $\mathfrak{m}$ (of size corresponding to $\lambda$) that if $(pk, sk) \leftarrow \mathsf{PKE.KeyGen}(1^\lambda)$, then

$$\mathsf{PKE.Dec}(sk, \mathsf{PKE.Enc}(pk, \mathfrak{m})) = \mathfrak{m},$$

except with negligible probability over the randomness used by $\mathsf{PKE.KeyGen}$, $\mathsf{PKE.Enc}$ and $\mathsf{PKE.Dec}$. The security notion we consider in this work is key dependent message security under chosen plaintext attacks. In the security experiment corresponding to this notion, the adversary gets a list of public keys $\{pk_i\}$ and access to an oracle that computes encryptions of functions of the secret keys. We call such dependencies *key cycles*, even though the functional relationships the adversary obtains can be more complex than key cycles.

**Definition 4.** *We say a public key encryption-scheme $\mathsf{PKE}$ is ciphertext indistinguishable under key dependent message chosen plaintext attacks (KDM-CPA) for cycles of length $l$ with respect to a class $\mathcal{F}$ of functions mapping $l$ secret keys to a plaintext, if every PPT-adversary $\mathcal{A}$ has success-probability at most negligibly better than $1/2$ in the experiment $\mathsf{KDM\text{-}CPA}_{\mathcal{F},l}$, i.e. $\Pr[\mathsf{KDM\text{-}CPA}_{\mathcal{F},l}(\mathcal{A}) = 1] \leq \frac{1}{2} + \mathsf{negl}(\lambda)$.*

**Experiment** $\mathsf{KDM\text{-}CPA}_{\mathcal{F},l}$ $\qquad \mathcal{O}_{\mathsf{KDM}}(f, j)$
$\quad$ For $i = 1, \ldots, l$ $\qquad\qquad\qquad$ If $b = 0$
$\qquad (pk_i, sk_i) \leftarrow \mathsf{PKE.KeyGen}(1^\lambda)$ $\qquad f \leftarrow \mathbf{0}$
$\quad b \leftarrow_{\$} \{0, 1\}$ $\qquad\qquad\qquad\qquad\quad \mathfrak{c} \leftarrow \mathsf{PKE.Enc}(pk_j, f(\{sk_i\}))$
$\quad b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot,\cdot)}(\{pk_i\}, 1^\lambda)$ $\qquad\quad$ Return $\mathfrak{c}$
$\quad$ Return 1 iff $b = b'$.

*Remark 1.* We implicitly assume that sanity checks are performed by the oracle, i.e. it only accepts KDM queries with $f \in \mathcal{F}$ and $j \in \{1, \ldots, l\}$. Moreover, we assume that the KDM oracle may have access to all local variables of the experiment $\mathsf{KDM\text{-}CPA}$, in particular the $pk_i$ and $sk_i$ and the bit $b$.

Applebaum [8] provides a general transfomation which transforms any public key encryption scheme with KDM security against *affine functions* into a public key encryption scheme with KDM security against arbitrary functions with circuits of *bounded size*. Thus, it is sufficient to construct a public key encryption scheme with KDM security against affine functions to obtain a scheme with security against the more general class of functions.

## 3 KDM Secure Public Key Encryption from Low Noise LPN

In this section we will provide a public key encryption scheme with KDM security for affine functions based on the hardness of the low noise LPN problem.

**Construction 1** *Let $n, m, k = \mathsf{poly}(\lambda)$ be positive integers with $m > k > n$. Let $\mathsf{C}$ be binary linear code of length $k$ and dimension $n$ and efficient encoding and decoding procedures $\mathsf{C.Encode}$ and $\mathsf{C.Decode}$. The public key encrypion scheme $\mathsf{PKE} = (\mathsf{PKE.KeyGen}, \mathsf{PKE.Enc}, \mathsf{PKE.Dec})$ is given by the following algorithms. The message space of $\mathsf{PKE}$ is $\mathbb{F}_2^n$.*

| $\mathsf{PKE.KeyGen}(1^\lambda)$: | $\mathsf{PKE.Enc}(pk, \mathfrak{m})$: | $\mathsf{PKE.Dec}(sk, \mathfrak{c})$: |
|---|---|---|
| $\mathbf{A} \leftarrow_\$ \mathbb{F}_2^{m \times n}$ | Parse $pk = (\mathbf{A}, \mathbf{b})$ | Parse $\mathfrak{c} = (\mathbf{C}_1, \mathbf{c}_2)$ and $sk = \mathbf{s}$ |
| $\mathbf{s} \leftarrow_\$ \mathbb{F}_2^n$ | $\mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$ | $\mathbf{z} \leftarrow \mathbf{c}_2 - \mathbf{C}_1 \mathbf{s}$ |
| $\mathbf{e} \leftarrow_\$ \mathsf{Ber}(m, \rho)$ | $\mathbf{C}_1 \leftarrow \mathbf{R} \cdot \mathbf{A}$ | $\mathfrak{m} \leftarrow \mathsf{C.Decode}(\mathbf{z})$ |
| $\mathbf{y} \leftarrow \mathbf{As} + \mathbf{e}$ | $\mathbf{c}_2 \leftarrow \mathbf{R} \cdot \mathbf{y} + \mathsf{C.Encode}(\mathfrak{m})$ | Return $\mathfrak{m}$ |
| $pk \leftarrow (\mathbf{A}, \mathbf{y})$ | $\mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$ | |
| $sk \leftarrow \mathbf{s}$ | Return $\mathfrak{c}$ | |
| Return $(pk, sk)$ | | |

### 3.1 Correctness

We will first show that the scheme $\mathsf{PKE}$ is correct.

**Lemma 4.** *Assume that $\mathsf{C.Decode}$ can efficiently decode from $\rho' = 4\rho^2 km$ errors. Then the scheme $\mathsf{PKE}$ is correct.*

The condition of Lemma 4 can be met by choosing $m, k = \Omega(n)$ and $\rho = O(1/\sqrt{n})$. Thus, to obtain conjectured $2^\lambda$-hardness for LPN, we can take usual parameter choices of $m, n, k = \Theta(\lambda^2)$ and $\rho = \Theta(1/\lambda)$ (as in [6,22,31]).

*Proof.* Assume that $\mathfrak{c} = (\mathbf{C}_1, \mathbf{c}_2)$ is a ciphertext generated by PKE.Enc. Consider the term $\mathbf{z}$ computed during decryption. It holds that

$$\begin{aligned}
\mathbf{z} &= \mathbf{c_2} - \mathbf{C}_1\mathbf{s} \\
&= \mathbf{Ry} + \text{C.Encode}(\mathfrak{m}) - \mathbf{RAs} \\
&= \text{C.Encode}(\mathfrak{m}) + \mathbf{R}(\mathbf{As} + \mathbf{e}) - \mathbf{RAs} \\
&= \text{C.Encode}(\mathfrak{m}) + \mathbf{Re}
\end{aligned}$$

By a Chernoff bound, it holds that $\|\mathbf{e}\| \leq 2\rho m$, except with negligible probability $e^{-\frac{1}{3}\rho m}$. Also by a Chernoff bound and a union bound, it holds that $\|\mathbf{R}\|_0 \leq 2\rho k$, except with negligible probability $m \cdot e^{-\frac{1}{3}\rho k}$. Therefore,

$$\|\mathbf{Re}\|_0 \leq \|\mathbf{R}\|_0 \cdot \|\mathbf{e}\|_0 \leq 4\rho^2 km,$$

except with negligible probability over the choice of $\mathbf{e}$ and $\mathbf{R}$. Consequently, C.Decode will be able to decode $\mathfrak{m}$ from $\mathbf{z}$.

### 3.2 KDM-CPA security

We will now prove KDM-CPA security of PKE.

**Theorem 1.** *Let $\lambda$ be a security parameter and $n, m, k, l = \text{poly}(\lambda)$ with $m \geq 2n$ and $l \geq 1$. Let $\rho = \rho(\lambda) \in [0, 1/2]$. Let $\mathcal{F} = \{f : (\mathbb{F}_2^n)^l \to \mathbb{F}_2^n\}$ be a family of affine functions. If $\text{eDLPN}(n, m, k, \rho, \text{Ber}(m, \rho))$, $\text{DLPN}(n, l \cdot m, \rho)$ and $\text{DDLPN}(n, m, k, \rho)$ are hard, then the scheme PKE is KDM-CPA$_{\mathcal{F},l}$ secure. More precisely, assume that $\mathcal{A}$ is a PPT adversary that breaks the KDM-CPA$_{\mathcal{F},l}$ security of PKE with advantage $\text{Adv}_{KDM\text{-}CPA}(\mathcal{A})$ and queries its KDM oracle at most $q = \text{poly}(\lambda)$ times. Then there exist PPT distinguishers $\mathcal{D}_1$ and $\mathcal{D}_2$ against the problem $\text{eDLPN}(n, m, k, \rho, \text{Ber}(m, \rho))$, $\mathcal{D}_3$ against $\text{DLPN}(n, l \cdot m, \rho)$ and $\mathcal{D}_4$ against $\text{DDLPN}(n, m, k, \rho)$ such that*

$$\begin{aligned}
\text{Adv}_{KDM\text{-}CPA}(\mathcal{A}) \leq{}& lq \cdot \text{Adv}_{eDLPN}(\mathcal{D}_1) + lq \cdot \text{Adv}_{eDLPN}(\mathcal{D}_2) \\
&+ \text{Adv}_{DLPN}(\mathcal{D}_3) + lq \cdot \text{Adv}_{DDLPN}(\mathcal{D}_4).
\end{aligned}$$

**Corollary 2.** *Let $n, m, k, l, \rho$ and $\mathcal{F}$ be as in Theorem 1. If $\text{LPN}(n, l \cdot m, \rho)$ is hard, then PKE is KDM-CPA$_{\mathcal{F},l}$ secure. More precisely, assume that $\mathcal{A}$ is a PPT adversary that breaks the KDM-CPA$_{\mathcal{F},l}$ of PKE with advantage $\epsilon$ and queries its KDM oracle at most $q = \text{poly}(\lambda)$ times. Then there exists a PPT adversary $\mathcal{A}^*$ that solves $\text{LPN}(n, l \cdot m, \rho)$ with advantage $\frac{\epsilon^2}{128k^2l^2q^2}$.*

The qualitative statement of Corollary 2 follows directly from Theorem 1 and Lemmas 1, 2 and 3. For the quantitative statement refer to the full version of this paper [21]. We will now provide a sketch for the proof of Theorem 1. See the full version [21] for the complete proof.

*Proof (Proof Sketch for Theorem 1).* Let $\mathcal{A}$ be a KDM-CPA adversary against PKE. Consider the following sequence of hybrid games. For notational convenience we assume that the oracles have access to all local variables of the games (without explicitly specifying so). We will first provide an overview of game 1 - 8 on the next pages.

**Game 1**

For $i = 1, \ldots, l$
$\quad \mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n}$
$\quad \mathbf{s}_i \leftarrow_\$ \mathbb{F}_2^n,\ \mathbf{e}_i \leftarrow_\$ \mathsf{Ber}(m, \rho)$
$\quad \mathbf{y} \leftarrow \mathbf{A}_i \mathbf{s}_i + \mathbf{e}_i$
$\quad pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i),\ sk \leftarrow \mathbf{s}_i$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
$\quad$ If $b = 0$
$\quad\quad f \leftarrow \mathbf{0}$
$\quad \mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$
$\quad \mathbf{C}_1 \leftarrow \mathbf{R} \cdot \mathbf{A}_j$
$\quad \mathbf{c}_2 \leftarrow \mathbf{R} \cdot \mathbf{y}_j + \mathsf{C.Encode}(f(\{sk_i\}))$
$\quad \mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
$\quad$ Return $\mathfrak{c}$

---

**Game 2**

$\boxed{\mathbf{s} \leftarrow \mathbb{F}_2^n}$
For $i = 1, \ldots, l$
$\quad \mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n},\ \mathbf{e}_i \leftarrow_\$ \mathsf{Ber}(m, \rho)$
$\quad \boxed{\mathbf{y}'_i \leftarrow \mathbf{A}_i \mathbf{s} + \mathbf{e}_i}$
$\quad \boxed{\mathbf{s}'_i \leftarrow_\$ \mathbb{F}_2^n}$
$\quad \boxed{\mathbf{s}_i \leftarrow \mathbf{s} + \mathbf{s}'_i,\ \mathbf{y}_i \leftarrow \mathbf{y}'_i + \mathbf{A}_i \mathbf{s}'_i}$
$\quad pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i),\ sk \leftarrow \mathbf{s}_i$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
$\quad$ If $b = 0$
$\quad\quad f \leftarrow \mathbf{0}$
$\quad \boxed{\text{Compute } \mathbf{T}_f, \mathbf{t}_f \text{ s.t. } f(\{sk_i\}) = \mathbf{T}_f \mathbf{s} + \mathbf{t}_f,}$
$\quad\quad \boxed{\text{using that } sk_i = \mathbf{s} + \mathbf{s}'_i}$
$\quad \mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$
$\quad \mathbf{C}_1 \leftarrow \mathbf{R} \cdot \mathbf{A}_j$
$\quad \boxed{\mathbf{c}_2 \leftarrow \mathbf{R} \cdot \mathbf{y}_j + \mathbf{G} \cdot (\mathbf{T}_f \mathbf{s} + \mathbf{t}_f)}$
$\quad \mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
$\quad$ Return $\mathfrak{c}$

---

**Game 3**

$\mathbf{s} \leftarrow \mathbb{F}_2^n$
For $i = 1, \ldots, l$
$\quad \mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n},\ \mathbf{e}_i \leftarrow_\$ \mathsf{Ber}(m, \rho)$
$\quad \mathbf{y}'_i \leftarrow \mathbf{A}_i \mathbf{s} + \mathbf{e}_i$
$\quad \mathbf{s}'_i \leftarrow_\$ \mathbb{F}_2^n$
$\quad \mathbf{s}_i \leftarrow \mathbf{s} + \mathbf{s}'_i,\ \mathbf{y}_i \leftarrow \mathbf{y}'_i + \mathbf{A}_i \mathbf{s}'_i$
$\quad pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i),\ sk \leftarrow \mathbf{s}_i$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$.

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
$\quad$ If $b = 0$
$\quad\quad f \leftarrow \mathbf{0}$
$\quad \text{Compute } \mathbf{T}_f, \mathbf{t}_f \text{ s.t. } f(\{sk_i\}) = \mathbf{T}_f \mathbf{s} + \mathbf{t}_f,$
$\quad\quad \text{using that } sk_i = \mathbf{s} + \mathbf{s}'_i$
$\quad \mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$
$\quad \mathbf{C}_1 \leftarrow \mathbf{R} \cdot \mathbf{A}_j$
$\quad \boxed{\mathbf{c}_2 \leftarrow (\mathbf{C}_1 + \mathbf{G}\mathbf{T}_f) \cdot \mathbf{s} + \mathbf{R}\mathbf{e}_j + \mathbf{G}\mathbf{t}_f + \mathbf{C}_1 \mathbf{s}'_j}$
$\quad \mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
$\quad$ Return $\mathfrak{c}$

---

**Game 4**

$\mathbf{s} \leftarrow \mathbb{F}_2^n$
For $i = 1, \ldots, l$
$\quad \mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n},\ \mathbf{e}_i \leftarrow_\$ \mathsf{Ber}(m, \rho)$
$\quad \mathbf{y}'_i \leftarrow \mathbf{A}_i \mathbf{s} + \mathbf{e}_i$
$\quad \mathbf{s}'_i \leftarrow_\$ \mathbb{F}_2^n$
$\quad \mathbf{s}_i \leftarrow \mathbf{s} + \mathbf{s}'_i,\ \mathbf{y}_i \leftarrow \mathbf{y}'_i + \mathbf{A}_i \mathbf{s}'_i$
$\quad pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i),\ sk \leftarrow \mathbf{s}_i$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$.

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
$\quad$ If $b = 0$
$\quad\quad f \leftarrow \mathbf{0}$
$\quad \text{Compute } \mathbf{T}_f, \mathbf{t}_f \text{ s.t. } f(\{sk_i\}) = \mathbf{T}_f \mathbf{s} + \mathbf{t}_f,$
$\quad\quad \text{using that } sk_i = \mathbf{s} + \mathbf{s}'_i$
$\quad \mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$
$\quad \boxed{\mathbf{U} \leftarrow_\$ \mathbb{F}_2^{k \times n}}$
$\quad \boxed{\mathbf{C}_1 \leftarrow \mathbf{U}}$
$\quad \mathbf{c}_2 \leftarrow (\mathbf{C}_1 + \mathbf{G}\mathbf{T}_f) \cdot \mathbf{s} + \mathbf{R}\mathbf{e}_j + \mathbf{G}\mathbf{t}_f + \mathbf{C}_1 \mathbf{s}'_j$
$\quad \mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
$\quad$ Return $\mathfrak{c}$

**Game 5**
$\mathbf{s} \leftarrow \mathbb{F}_2^n$
For $i = 1, \ldots, l$
  $\mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n}$, $\mathbf{e}_i \leftarrow_\$ \mathsf{Ber}(m, \rho)$
  $\mathbf{y}_i' \leftarrow \mathbf{A}_i \mathbf{s} + \mathbf{e}_i$
  $\mathbf{s}_i' \leftarrow_\$ \mathbb{F}_2^n$
  $\mathbf{s}_i \leftarrow \mathbf{s} + \mathbf{s}_i'$, $\mathbf{y}_i \leftarrow \mathbf{y}_i' + \mathbf{A}_i \mathbf{s}_i'$
  $pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i)$, $sk \leftarrow \mathbf{s}_i$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$.

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
  If $b = 0$
    $f \leftarrow \mathbf{0}$
  Compute $\mathbf{T}_f, \mathbf{t}_f$ s.t. $f(\{sk_i\}) = \mathbf{T}_f \mathbf{s} + \mathbf{t}_f$,
    using that $sk_i = \mathbf{s} + \mathbf{s}_i'$
  $\mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$
  $\mathbf{U} \leftarrow_\$ \mathbb{F}_2^{k \times n}$
  $\boxed{\mathbf{C}_1 \leftarrow \mathbf{U} - \mathbf{G}\mathbf{T}_f}$
  $\boxed{\mathbf{c}_2 \leftarrow \mathbf{U}\mathbf{s} + \mathbf{R}\mathbf{e}_j + \mathbf{G}\mathbf{t}_f + \mathbf{C}_1 \mathbf{s}_j'}$
  $\mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
  Return $\mathfrak{c}$

---

**Game 6**
$\mathbf{s} \leftarrow \mathbb{F}_2^n$
For $i = 1, \ldots, l$
  $\mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n}$, $\mathbf{e}_i \leftarrow_\$ \mathsf{Ber}(m, \rho)$
  $\mathbf{y}_i' \leftarrow \mathbf{A}_i \mathbf{s} + \mathbf{e}_i$
  $\mathbf{s}_i' \leftarrow_\$ \mathbb{F}_2^n$
  $\mathbf{s}_i \leftarrow \mathbf{s} + \mathbf{s}_i'$, $\mathbf{y}_i \leftarrow \mathbf{y}_i' + \mathbf{A}_i \mathbf{s}_i'$
  $pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i)$, $sk \leftarrow \mathbf{s}_i$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$.

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
  If $b = 0$
    $f \leftarrow \mathbf{0}$
  Compute $\mathbf{T}_f, \mathbf{t}_f$ s.t. $f(\{sk_i\}) = \mathbf{T}_f \mathbf{s} + \mathbf{t}_f$,
    using that $sk_i = \mathbf{s} + \mathbf{s}_i'$
  $\mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$
  $\boxed{\mathbf{C}_1 \leftarrow \mathbf{R}\mathbf{A}_j - \mathbf{G}\mathbf{T}_f}$
  $\boxed{\mathbf{c}_2 \leftarrow \mathbf{R}\mathbf{A}_j \mathbf{s} + \mathbf{R}\mathbf{e}_j + \mathbf{G}\mathbf{t}_f + \mathbf{C}_1 \mathbf{s}_j'}$
  $\mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
  Return $\mathfrak{c}$

---

**Game 7**
For $i = 1, \ldots, l$
  $\mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n}$
  $\boxed{\mathbf{y}_i' \leftarrow_\$ \mathbb{F}_2^m}$
  $\mathbf{s}_i' \leftarrow_\$ \mathbb{F}_2^n$
  $\mathbf{y}_i \leftarrow \mathbf{y}_i' + \mathbf{A}_i \mathbf{s}_i'$
  $pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i)$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$.

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
  If $b = 0$
    $f \leftarrow \mathbf{0}$
  Compute $\mathbf{T}_f, \mathbf{t}_f$ s.t. $f(\{sk_i\}) = \mathbf{T}_f \mathbf{s} + \mathbf{t}_f$,
    using that $sk_i = \mathbf{s} + \mathbf{s}_i'$
  $\mathbf{R} \leftarrow_\$ \mathsf{Ber}(k \times m, \rho)$
  $\mathbf{C}_1 \leftarrow \mathbf{R}\mathbf{A}_j - \mathbf{G}\mathbf{T}_f$
  $\boxed{\mathbf{c}_2 \leftarrow \mathbf{R}\mathbf{y}_j' + \mathbf{G}\mathbf{t}_f + \mathbf{C}_1 \mathbf{s}_j'}$
  $\mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
  Return $\mathfrak{c}$

---

**Game 8**
For $i = 1, \ldots, l$
  $\mathbf{A}_i \leftarrow_\$ \mathbb{F}_2^{m \times n}$
  $\mathbf{y}_i' \leftarrow_\$ \mathbb{F}_2^m$
  $\mathbf{s}_i' \leftarrow_\$ \mathbb{F}_2^n$
  $\mathbf{y}_i \leftarrow \mathbf{y}_i' + \mathbf{A}_i \mathbf{s}_i'$
  $pk_i \leftarrow (\mathbf{A}_i, \mathbf{y}_i)$
$b \leftarrow_\$ \{0, 1\}$
$b' \leftarrow \mathcal{A}^{\mathcal{O}_{\mathsf{KDM}}(\cdot, \cdot)}(\{pk_i\}, 1^\lambda)$
Return 1 iff $b = b'$.

$\mathcal{O}_{\mathsf{KDM}}(f, j)$
  $\boxed{\mathbf{U} \leftarrow_\$ \mathbb{F}_2^{k \times n}}$
  $\boxed{\mathbf{u} \leftarrow_\$ \mathbb{F}_2^k}$
  $\boxed{\mathbf{C}_1 \leftarrow \mathbf{U}}$
  $\boxed{\mathbf{c}_2 \leftarrow \mathbf{u}}$
  $\mathfrak{c} \leftarrow (\mathbf{C}_1, \mathbf{c}_2)$
  Return $\mathfrak{c}$

- **Game 1** is identical to the KDM-CPA experiment, we only replace the algorithms PKE.KeyGen and PKE.Enc with their instantiations according to PKE.
- In **game 2**, we change the experiment in three ways. First, the public and secret keys are computed from a *master secret* $\mathbf{s}$. More specifically, we first choose $\mathbf{s} \leftarrow_\$ \mathbb{F}_2^n$ uniformly at random and then compute $\mathbf{y}_i' \leftarrow \mathbf{A}_i\mathbf{s} + \mathbf{e}_i$ for each index $i$. We obtain the public and secret keys by rerandomizing $\mathbf{s}$ and $\mathbf{y}_i'$ correlated way. Specifically, we set $\mathbf{s}_i = \mathbf{s} + \mathbf{s}_i'$ and $\mathbf{y}_i = \mathbf{y}_i' + \mathbf{A}_i\mathbf{s}_i'$ for a uniformly random and independent $\mathbf{s}_i'$. Since $\mathbf{s}_i$ is uniformly random and independent of $\mathbf{s}$ and further

$$\mathbf{y}_i = \mathbf{y}_i' + \mathbf{A}_i\mathbf{s}_i' = \mathbf{A}_i\mathbf{s}_i + \mathbf{e}_i,$$

we have that the $(pk_i, sk_i)$ are identically distributed to game 1. Thus, this modification to the experiment did not introduce any statistically difference. Secondly, we write $\mathsf{C.Encode}(\cdot)$ using the generator matrix $\mathbf{G}$ of $\mathsf{C}$, which is also merely a syntactical change.

The third modification consists in changing the way the functions $f$ the encryption oracle is queried with are evaluated. Since each $f$ is restricted to be an affine function, we can write it as

$$f(\{sk_i\}) = \sum_{i=1}^{l} \mathbf{T}_i\mathbf{s}_i + \mathbf{t}.$$

for $\mathbf{T}_1, ,\ldots, \mathbf{T}_l \in \mathbb{F}_2^{n \times n}$ and $\mathbf{t} \in \mathbb{F}_2^n$. Using that $\mathbf{s}_i = \mathbf{s} + \mathbf{s}_i'$ we can write

$$f(\{sk_i\}) = \sum_{i=1}^{l} \mathbf{T}_i(\mathbf{s} + \mathbf{s}_i') + \mathbf{t}$$

$$= \left(\sum_{i=1}^{l} \mathbf{T}_i\right)\mathbf{s} + \sum_{i=1}^{l} \mathbf{T}_i\mathbf{s}_i' + \mathbf{t}.$$

Therefore, setting $\mathbf{T}_f = \sum_{i=1}^{l} \mathbf{T}_i$ and $\mathbf{t}_f = \sum_{i=1}^{l} \mathbf{T}_i\mathbf{s}_i' + \mathbf{t}$ we can write $f(\{sk_i\}) = \mathbf{T}_f\mathbf{s} + \mathbf{t}_f$. Thus, also the third modification does not introduce any statistical difference.
- In **game 3** we change the way $\mathbf{c}_2$ is computed. However, plugging in $\mathbf{y}_j = \mathbf{A}_j\mathbf{s}_j + \mathbf{e}_j = \mathbf{A}_j(\mathbf{s} + \mathbf{s}_j') + \mathbf{e}_j$ and rearranging terms yields

$$\begin{aligned}
\mathbf{c}_2 &= \mathbf{R} \cdot \mathbf{y}_j + \mathbf{G} \cdot (\mathbf{T}_f\mathbf{s} + \mathbf{t}_f) \\
&= \mathbf{R} \cdot (\mathbf{A}_j(\mathbf{s} + \mathbf{s}_j') + \mathbf{e}_j) + \mathbf{G} \cdot (\mathbf{T}_f\mathbf{s} + \mathbf{t}_f) \\
&= (\mathbf{R}\mathbf{A} + \mathbf{G}\mathbf{T}_f)\mathbf{s} + \mathbf{R}\mathbf{e}_j + \mathbf{G}\mathbf{t}_f + \mathbf{R}\mathbf{A}\mathbf{s}_j' \\
&= (\mathbf{C}_1 + \mathbf{G}\mathbf{T}_f)\mathbf{s} + \mathbf{R}\mathbf{e}_j + \mathbf{G}\mathbf{t}_f + \mathbf{C}_1\mathbf{s}_j'.
\end{aligned}$$

- In **game 4**, at every call to $\mathcal{O}_{\mathsf{KDM}}$ the value $\mathbf{C}_1$ is chosen uniformly at random instead of computed by $\mathbf{C}_1 \leftarrow \mathbf{R}\mathbf{A}_j$. We can show that game 3 and game 4 are computationally indistinguishable, given that the problem $\mathsf{eDLPN}(n, m, k, \rho, \mathsf{Ber}(m, \rho))$ is hard. The reduction loses a factor if $lq$.

- In **game 5** we compute $\mathbf{C}_1$ by $\mathbf{C}_1 \leftarrow \mathbf{U} - \mathbf{GT}_f$. Since $\mathbf{U}$ is chosen independently and uniformly at random, **game 4** and **game 5** are identically distributed from the view of $\mathcal{A}$.
- In **game 6** we replace $\mathbf{U}$ again by $\mathbf{RA}_j$. We can show that game 5 and game 6 are computationally indistinguishable provided that the problem $\mathsf{eDLPN}(n, m, k, \rho, \mathsf{Ber}(m, \rho))$ is hard. The reduction loses a factor if $lq$.
- In **game 7** we choose the $\mathbf{y}_i'$ uniformly at random instead of by $\mathbf{y}_i' \leftarrow \mathbf{A}_i \mathbf{s} + \mathbf{e}_i$. We can show that game 6 and game 7 are computationally indistinguishable, given that $\mathsf{DLPN}(n, l \cdot m, \rho)$ is hard. This reduction is tight.
- In **game 8** the values $\mathbf{C}_1$ and $\mathbf{c}_2$ are chosen uniformly at random. Therefore the output of $\mathcal{O}_{\mathsf{KDM}}$ is independent of the challenge bit $b$ and consequently $\mathcal{A}$'s advantage in game 8 is 0. We can show that game 7 and game 8 are computationally indistinguishable given that $\mathsf{DDLPN}(n, m, k, \rho)$ is hard. This reduction loses a factor of $lq$.

## 4    LPN Sample Amplification

In this Section, we will show that the hardness of LPN with a bounded number of samples implies the hardness of LPN with an unbounded number of samples, if one is willing to accept an increase in the amount of noise. Recall that $\mathsf{S}(2n, \rho)$ is the uniform distribution on vectors of weight $\lfloor \rho 2n \rfloor$ in $\mathbb{F}_2^n$.

**Theorem 2.** *Let $\lambda$ be a security parameter, $n = \mathsf{poly}(\lambda)$ be a positive integer and $\rho = \rho(\lambda) \in [0, 1/2]$. Let*

$$\rho' \geq \frac{1}{2}(1 - (1 - 2\rho)^{\lfloor \rho 2n \rfloor}).$$

*If $\mathsf{eDLPN}(n, 2n, 1, \rho, \mathsf{S}(2n, \rho))$, $\mathsf{DLPN}(n, 2n, \mathsf{S}(2n, \rho))$ and $\mathsf{DDLPN}(n, 2n, 1, \rho)$ are hard, then it holds that $\mathsf{DLPN}(n, \rho')$ is also hard. Precisely, if $\mathcal{D}$ is a PPT distinguisher against $\mathsf{DLPN}(n, \rho')$ that makes at most $q$ queries to its LPN oracle, then there exist PPT distinguishers $\mathcal{D}_1$ against $\mathsf{eDLPN}(n, 2n, 1, \rho, \mathsf{S}(2n, \rho))$, $\mathcal{D}_2$ against $\mathsf{DLPN}(n, 2n, \mathsf{S}(2n, \rho))$ and $\mathcal{D}_3$ against $\mathsf{DDLPN}(n, 2n, 1, \rho)$ such that*

$$\mathsf{Adv}_{\mathsf{DLPN}}(\mathcal{D}) \leq q \cdot \mathsf{Adv}_{\mathsf{eDLPN}}(\mathcal{D}_1) + \mathsf{Adv}_{\mathsf{DLPN}}(\mathcal{D}_2) + q \cdot \mathsf{Adv}_{\mathsf{DDLPN}}(\mathcal{D}_3).$$

**Corollary 3.** *Let $n$, $\rho$ and $\rho'$ be as in Theorem 2. If $\mathsf{LPN}(n, 2n, \rho)$ is hard, then $\mathsf{DLPN}(n, \rho')$ is also hard. More precisely, if $\mathcal{D}$ is a PPT distinguisher which distinguishes $\mathsf{DLPN}(n, \rho')$ with advantage $\epsilon$ and makes at most $q$ queries to its LPN oracle, then there exists a PPT adversary $\mathcal{A}^*$ which breaks $\mathsf{LPN}(n, 2n, \rho)$ with advantage*

$$\mathsf{Adv}_{\mathsf{LPN}}(\mathcal{A}^*) \geq \frac{\epsilon^2}{72q^2}.$$

The qualitative statement of Corollary 3 follows immediately from Theorem 2 and Lemmas 1, 2 and 3. For the quantitative part we refer to the full version of this paper [21].

Corollary 3 can be seen as a trade-off between noise and extra samples. We tolerate that the amount of noise required gets squared, while in turn we get an arbitrary polynomial amount of samples.

*Proof (Proof of Theorem 2).* We will prove the theorem for the minimal $\rho'$, i.e. $\rho' = \frac{1}{2}(1 - (1 - 2\rho)^{\lfloor \rho 2n \rfloor})$. Let $\mathcal{D}$ be a PPT distinguisher against $\mathsf{DLPN}(n, \rho')$. We will provide a series of hybrid experiments $\mathsf{Exp}_1, \mathsf{Exp}_2, \mathsf{Exp}_3, \mathsf{Exp}_4$ and show that from the view of $\mathcal{D}$ any two of experiments the are indistinguishable. We will provide the experiments by defining the sample oracles $\mathcal{O}$ the distinguisher $\mathcal{D}$ gets access to.

| **Experiment** $\mathsf{Exp}_1$ | **Experiment** $\mathsf{Exp}_2$ | **Experiment** $\mathsf{Exp}_3$ | **Experiment** $\mathsf{Exp}_4$ |
|---|---|---|---|
| Initialization: | Initialization: | Initialization: | Initialization: |
| $\mathbf{s} \leftarrow_{\$} \mathbb{F}_2^n$ | $\mathbf{A} \leftarrow_{\$} \mathbb{F}_2^{2n \times n}$ | $\mathbf{A} \leftarrow_{\$} \mathbb{F}_2^{2n \times n}$ | - |
| Oracle $\mathcal{O}_{\mathsf{Exp}_1}()$ | $\mathbf{s} \leftarrow_{\$} \mathbb{F}_2^n$ | $\mathbf{r} \leftarrow_{\$} \mathbb{F}_2^{2n}$ | Oracle $\mathcal{O}_{\mathsf{Exp}_4}()$ |
| $\mathbf{a} \leftarrow_{\$} \mathbb{F}_2^n$ | $\mathbf{z} \leftarrow_{\$} \mathsf{S}(2n, \rho)$ | Oracle $\mathcal{O}_{\mathsf{Exp}_3}()$ | $\mathbf{a} \leftarrow_{\$} \mathbb{F}_2^n$ |
| $e \leftarrow_{\$} \mathsf{Ber}(\rho')$ | $\mathbf{r} \leftarrow \mathbf{As} + \mathbf{z}$ | $\mathbf{e} \leftarrow_{\$} \mathsf{Ber}(2n, \rho)$ | $y \leftarrow_{\$} \mathbb{F}_2$ |
| $y \leftarrow \langle \mathbf{a}, \mathbf{s} \rangle + e$ | Oracle $\mathcal{O}_{\mathsf{Exp}_2}()$ | $\mathbf{a} \leftarrow \mathbf{e}^\top \mathbf{A}$ | Return $(\mathbf{a}, y)$ |
| Return $(\mathbf{a}, y)$ | $\mathbf{e} \leftarrow_{\$} \mathsf{Ber}(2n, \rho)$ | $y \leftarrow \langle \mathbf{e}, \mathbf{r} \rangle$ | |
| | $\mathbf{a} \leftarrow \mathbf{e}^\top \mathbf{A}$ | Return $(\mathbf{a}, y)$ | |
| | $y \leftarrow \langle \mathbf{e}, \mathbf{r} \rangle$ | | |
| | Return $(\mathbf{a}, y)$ | | |

Clearly, experiment $\mathsf{Exp}_1$ provides samples from the LPN distribution while experiment $\mathsf{Exp}_4$ provides uniformly random samples. Thus, we need to establish that from the view of $\mathcal{D}$ the experiments $\mathsf{Exp}_1$ and $\mathsf{Exp}_4$ are indistinguishable. We will start with the indistinguishability of $\mathsf{Exp}_1$ and $\mathsf{Exp}_2$. Assume that $\mathcal{D}$ distinguishes with advantage $\epsilon_1$ between $\mathsf{Exp}_1$ and $\mathsf{Exp}_2$, i.e.

$$|\Pr[\mathsf{Exp}_1(\mathcal{D}) = 1] - \Pr[\mathsf{Exp}_2(\mathcal{D}) = 1]| = \epsilon_1.$$

Assume further that $q = \mathsf{poly}(\lambda)$ is an upper bound on the number of samples $\mathcal{D}$ queries. We will construct a PPT distinguisher $\mathcal{D}_1$ that distinguishes the problem $\mathsf{eDLPN}(n, 2n, 1, \rho, \mathsf{S}(2n, \rho))$ with advantage $\geq \epsilon_1/q$. The distinguisher $\mathcal{D}_1$ is given on the left side of Figure 2.

Notice that $\mathcal{D}_1$ answers the first $i^* - 1$ oracle queries of $\mathcal{D}$ exactly like $\mathsf{Exp}_1$, while it answers the last $q - i^*$ queries like $\mathsf{Exp}_2$. In the $i^*$-th query however, $\mathcal{D}_1$ embeds its own challenge. Moreover, notice that $\mathcal{D}_1$ is efficient as $\mathcal{D}$ is efficient. To analyze the distinguishing advantage of $\mathcal{D}_1$, we will define a sequence of hybrid experiments $\mathsf{H}_0, \ldots, \mathsf{H}_q$. $\mathsf{H}_i$ is crafted to answer the first $i$ queries like $\mathsf{Exp}_1$, while it answers the last $q - i$ queries like $\mathsf{Exp}_2$. Experiment $\mathsf{H}_i$ is given on the right side of Figure 2.

We are now ready to analyze the distinguishing advantage of $\mathcal{D}_1$. First assume that $\mathcal{D}_1$'s input is of the form $(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{z}, \mathbf{e}^\top \mathbf{z} = \langle \mathbf{z}, \mathbf{e} \rangle)$. Observe that since $\mathbf{z}$ has

| **Distinguisher** $\mathcal{D}_1$ | Oracle $\mathcal{O}_{\mathcal{D}_1}()$ | **Experiment** $\mathsf{H}_i$ | Oracle $\mathcal{O}_{\mathsf{H}_i}()$ |
|---|---|---|---|
| Input: $(\mathbf{A}, \mathbf{c}, \mathbf{z}, t)$ | If $cnt < i^*$ | Initialization: | If $cnt \le i$ |
| $i^* \leftarrow_\$ \{1, \ldots, q\}$ | $\quad \mathbf{a} \leftarrow_\$ \mathbb{F}_2^n$ | $\quad \mathbf{A} \leftarrow_\$ \mathbb{F}_2^{2n \times n}$ | $\quad \mathbf{a} \leftarrow_\$ \mathbb{F}_2^n$ |
| $\mathbf{s} \leftarrow_\$ \mathbb{F}_2^n$ | $\quad e \leftarrow_\$ \mathsf{Ber}(\rho')$ | $\quad \mathbf{s} \leftarrow_\$ \mathbb{F}_2^n$ | $\quad e \leftarrow_\$ \mathsf{Ber}(\rho')$ |
| $\mathbf{r} = \mathbf{As} + \mathbf{z}$ | $\quad y \leftarrow \langle \mathbf{a}, \mathbf{s}\rangle + e$ | $\quad \mathbf{z} \leftarrow_\$ \mathsf{S}_{2n}(\lfloor \rho 2n\rfloor)$ | $\quad y \leftarrow \langle \mathbf{a}, \mathbf{s}\rangle + e$ |
| $cnt = 1$ | If $cnt = i^*$ | $\quad \mathbf{r} \leftarrow \mathbf{As} + \mathbf{z}$ | If $cnt > i$ |
| $b \leftarrow \mathcal{D}^{\mathcal{O}_{\mathcal{D}_1}()}$ | $\quad \mathbf{a} \leftarrow \mathbf{c}^\top$ | $\quad cnt \leftarrow 1$ | $\quad \mathbf{e} \leftarrow_\$ \mathsf{Ber}(2n, \rho)$ |
| return $b$ | $\quad y \leftarrow \langle \mathbf{c}, \mathbf{s}\rangle + t$ | | $\quad \mathbf{a} \leftarrow (\mathbf{e}^\top \mathbf{A})^\top$ |
| | If $cnt > i^*$ | | $\quad y \leftarrow \langle \mathbf{e}, \mathbf{r}\rangle$ |
| | $\quad \mathbf{e} \leftarrow_\$ \mathsf{Ber}(2n, \rho)$ | | $cnt \leftarrow cnt + 1$ |
| | $\quad \mathbf{a} \leftarrow (\mathbf{e}^\top \mathbf{A})^\top$ | | Return $(\mathbf{a}, y)$ |
| | $\quad y \leftarrow \langle \mathbf{e}, \mathbf{r}\rangle$ | | |
| | $cnt \leftarrow cnt + 1$ | | |
| | Return $(\mathbf{a}, y)$ | | |

**Fig. 2.** The distinguisher $\mathcal{D}_1$ and the hybrid experiments $\mathsf{H}_i$

weight $\lfloor \rho m\rfloor$ and $\mathbf{e}$ is distributed according to $\mathsf{Ber}(m, \rho)$ it holds that $\langle \mathbf{z}, \mathbf{e}\rangle$ is distributed according to $\mathsf{Ber}(\rho')$. Fix the random choice $i^* = i$. Then the sample oracle $\mathcal{O}_{\mathcal{D}_1}()$ implemented by $\mathcal{D}_1$ behaves identical to the sample oracle of $\mathsf{H}_{i-1}$. Consequently, it holds that

$$\Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1 | i^* = i] = \Pr[\mathsf{H}_{i-1}(\mathcal{D}) = 1]$$

and thus, as $i^*$ is uniformly chosen from $\{1, \ldots, q\}$

$$\Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1] = \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1 | i^* = i]$$

$$= \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathsf{H}_{i-1}(\mathcal{D}) = 1].$$

Next assume that $\mathcal{D}_1$'s input is of the form $(\mathbf{A}, \mathbf{u}, \mathbf{z}, \mathbf{e}^\top \mathbf{z})$. Again, fix the random choice of $i^*$ to $i^* = i$. Then the sample oracle $\mathcal{O}_{\mathcal{D}_1}()$ implemented by $\mathcal{D}_1$ behaves identical to the sample oracle of $\mathsf{H}_i$, as $\langle \mathbf{z}, \mathbf{e}\rangle$ is distributed according to $\mathsf{Ber}(\rho')$. Consequently,

$$\Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{u}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1 | i^* = i] = \Pr[\mathsf{H}_i(\mathcal{D}) = 1]$$

and thus

$$\Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{u}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1] = \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{u}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1 | i^* = i]$$

$$= \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathsf{H}_i(\mathcal{D}) = 1].$$

Together, this yields

$$\mathsf{Adv}_{\mathsf{eDLPN}}(\mathcal{D}_1) = |\Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1] - \Pr[\mathcal{D}_1(\mathbf{A}, \mathbf{u}, \mathbf{z}, \mathbf{e}^\top \mathbf{z}) = 1]|$$

$$= \left| \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathsf{H}_{i-1}(\mathcal{D}) = 1] - \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathsf{H}_i(\mathcal{D}) = 1] \right|$$

$$= \frac{1}{q} |\Pr[\mathsf{H}_0(\mathcal{D}) = 1] - \Pr[\mathsf{H}_k(\mathcal{D}) = 1]|$$

$$= \frac{1}{q} |\Pr[\mathsf{Exp}_2(\mathcal{D}) = 1] - \Pr[\mathsf{Exp}_1(\mathcal{D}) = 1]|$$

$$\geq \epsilon_1/q.$$

Thus, $\mathcal{D}_1$ distinguishes $\mathsf{eDLPN}(n, 2n, 1, \rho, \mathsf{S}(2n, \rho))$ with advantage $\epsilon_1/q$.

Next, we turn to the indistinguishability of $\mathsf{Exp}_2$ and $\mathsf{Exp}_3$. Assume towards contradiction that $\mathcal{D}$ distinguishes between $\mathsf{Exp}_2$ and $\mathsf{Exp}_3$ with advantage $\epsilon_2$, i.e.

$$|\Pr[\mathsf{Exp}_2(\mathcal{D}) = 1] - \Pr[\mathsf{Exp}_3(\mathcal{D}) = 1]| = \epsilon_2.$$

We will construct a PPT distinguisher $\mathcal{D}_2$ against $\mathsf{DLPN}(n, 2n, \mathsf{S}(2n, \rho)$. $\mathcal{D}_2$ is given as follows.

| **Distinguisher** $\mathcal{D}_2$ | Sample Oracle $\mathcal{O}_{\mathcal{D}_2}()$ |
|---|---|
| Input: $(\mathbf{A}, \mathbf{r})$ | $\mathbf{e} \leftarrow_\$ \mathsf{Ber}(2n, \rho)$ |
| $b \leftarrow \mathcal{D}^{\mathcal{O}_{\mathcal{D}_2}()}$ | $\mathbf{a} \leftarrow \mathbf{e}^\top \mathbf{A}$ |
| return $b$ | $y \leftarrow \langle \mathbf{e}, \mathbf{r} \rangle$ |
| | Return $(\mathbf{a}, y)$ |

The distinguisher $\mathcal{D}_2$ is efficient, as $\mathcal{D}$ is efficient. First, assume that $\mathcal{D}_2$'s input is of the form $(\mathbf{A}, \mathbf{As} + \mathbf{z})$, where $\mathbf{s}$ is chosen uniformly from $\mathbb{F}_2^n$ and $\mathbf{z}$ is chosen from $\mathsf{S}(2n, \rho)$. Then clearly the sample oracle $\mathcal{O}_{\mathcal{D}_2}$ behaves just as in $\mathsf{Exp}_2$. On the other hand, if $\mathcal{D}_2$'s input is of the form $(\mathbf{A}, \mathbf{u})$ with $\mathbf{u}$ chosen uniformly random from $\mathbb{F}_2^{2n}$, then the sample $\mathcal{O}_{\mathcal{D}_2}$ simulated by $\mathcal{D}_2$ behaves like the sample oracle in $\mathsf{Exp}_3$. Consequently, it holds that

$$\mathsf{Adv}_{\mathsf{DLPN}}(\mathcal{D}_2) = |\Pr[\mathcal{D}_2(\mathbf{A}, \mathbf{As} + \mathbf{z}) = 1] - \Pr[\mathcal{D}_2(\mathbf{A}, \mathbf{u}) = 1]|$$

$$= |\Pr[\mathsf{Exp}_2(\mathcal{D}) = 1] - \Pr[\mathsf{Exp}_3(\mathcal{D}) = 1]|$$

$$= \epsilon_2.$$

Thus, the distinguishing advantage of $\mathcal{D}_2$ against $\mathsf{DLPN}(n, 2n, \mathsf{S}(2n, \rho))$ is $\epsilon_2$.

We will finally turn to showing that from the view of $\mathcal{D}$, $\mathsf{Exp}_3$ and $\mathsf{Exp}_4$ are indistinguishable. Assume towards contradiction that $\mathcal{D}$ distinguishes between $\mathsf{Exp}_3$ and $\mathsf{Exp}_4$ with advantage $\epsilon_3$, i.e.

$$|\Pr[\mathsf{Exp}_3(\mathcal{D}) = 1] - \Pr[\mathsf{Exp}_4(\mathcal{D}) = 1]| = \epsilon_3.$$

Assume further $\mathcal{D}$ makes at most $q = \mathsf{poly}(\lambda)$ queries to its sample oracle. We will construct a PPT distinguisher $\mathcal{D}_3$ that distinguishes $\mathsf{DDLPN}(n, 2n, 1, \rho)$ with advantage $\epsilon_3/q$. The distinguisher $\mathcal{D}_3$ is given on the left side of Figure 3.

**Distinguisher $\mathcal{D}_3$** | Oracle $\mathcal{O}_{\mathcal{D}_3}()$ | **Experiment $\mathsf{H}'_i$** | Oracle $\mathcal{O}_{\mathsf{H}'_i}()$
--- | --- | --- | ---
Input: $(\mathbf{A}, \mathbf{a}^*, \mathbf{r}, y^*)$ | If $cnt < i^*$ | Initialization: | If $cnt \leq i$
$i^* \leftarrow_\$ \{1, \ldots, q\}$ | $\quad \mathbf{e} \leftarrow_\$ \mathsf{Ber}(2n, \rho)$ | $\quad \mathbf{A} \leftarrow_\$ \mathbb{F}_2^{2n \times n}$ | $\quad \mathbf{e} \leftarrow_\$ \mathsf{Ber}(2n, \rho)$
$cnt = 1$ | $\quad \mathbf{a} \leftarrow (\mathbf{e}^\top \mathbf{A})^\top$ | $\quad \mathbf{r} \leftarrow_\$ \mathbb{F}_2^{2n}$ | $\quad \mathbf{a} \leftarrow (\mathbf{e}^\top \mathbf{A})^\top$
$b \leftarrow \mathcal{D}^{\mathcal{O}_{\mathcal{D}_3}}()$ | $\quad y \leftarrow \langle \mathbf{e}, \mathbf{r} \rangle$ | $\quad cnt \leftarrow 1$ | $\quad y \leftarrow \langle \mathbf{e}, \mathbf{r} \rangle$
return $b$ | If $cnt = i^*$ | | If $cnt > i$
| $\quad \mathbf{a} \leftarrow \mathbf{a}^{*\top}$ | | $\quad \mathbf{a} \leftarrow_\$ \mathbb{F}_2^n$
| $\quad y \leftarrow y^*$ | | $\quad y \leftarrow_\$ \mathbb{F}_2$
| If $cnt > i^*$ | | $cnt \leftarrow cnt + 1$
| $\quad \mathbf{a} \leftarrow_\$ \mathbb{F}_2^n$ | | Return $(\mathbf{a}, y)$
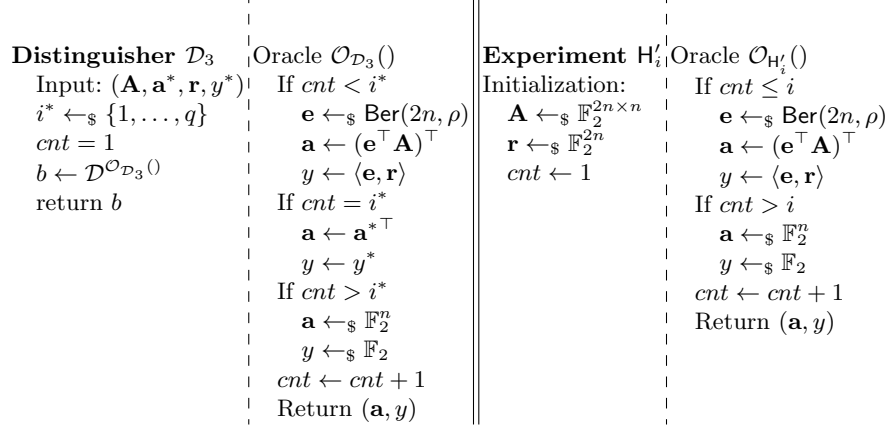| $\quad y \leftarrow_\$ \mathbb{F}_2$ | |
| $cnt \leftarrow cnt + 1$ | |
| Return $(\mathbf{a}, y)$ | |

**Fig. 3.** The distinguisher $\mathcal{D}_3$ and the hybrid experiments $\mathsf{H}'_i$

It is clear that $\mathcal{D}_3$ is efficient, once $\mathcal{D}$ is efficient. Again, to analyze the distinguishing advantage of $\mathcal{D}_3$, we will define a sequence of hybrid experiments $\mathsf{H}'_0, \ldots, \mathsf{H}'_q$. $\mathsf{H}'_i$ is crafted to answer the first $i$ queries like $\mathsf{Exp}_3$, while it answers the last $q - i$ queries like $\mathsf{Exp}_4$. Hybrid $\mathsf{H}'_i$ is given on the right side of Figure 3. First assume that $\mathcal{D}_3$'s input is of the form $(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{r}, \mathbf{e}^\top \mathbf{r})$. Then it holds that

$$\mathbf{a}^* = (\mathbf{e}^\top \mathbf{A})^\top$$
$$y^* = \mathbf{e}^\top \mathbf{r} = \langle \mathbf{e}, \mathbf{r} \rangle$$

Now fix a random choice $i^* = i$. Then $\mathcal{O}_{\mathcal{D}_3}()$ in $\mathcal{D}_3$'s simulation behaves identically to the sample oracle in $\mathsf{H}'_i$. Thus it holds that

$$\Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{r}, \mathbf{e}^\top \mathbf{r}) = 1 | i^* = i] = \Pr[\mathsf{H}'_i(\mathcal{D}) = 1],$$

and consequently

$$\Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{r}, \mathbf{e}^\top \mathbf{r}) = 1] = \sum_{i=1}^{q} \frac{1}{q} \Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{r}, \mathbf{e}^\top \mathbf{r}) = 1 | i^* = i]$$
$$= \sum_{i=1}^{q} \frac{1}{q} \Pr[\mathsf{H}'_i(\mathcal{D}) = 1].$$

Now suppose that $\mathcal{D}_3$'s input is of the form $(\mathbf{A}, \mathbf{u}, \mathbf{r}, u')$, where $\mathbf{u}^\top \leftarrow_\$ \mathbb{F}_2^n$ and $u' \leftarrow_\$ \mathbb{F}_2$ are chosen uniformly at random. Then it holds that $\mathbf{a}^* = \mathbf{u}$ and

$y^* = u'$. Again, fix a random choice $i^* = i$. Then $\mathcal{O}_{\mathcal{D}_3}()$ in $\mathcal{D}_3$'s simulation behaves identically to the sample oracle in $\mathsf{H}'_{i-1}$. Thus it holds that

$$\Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{u}, \mathbf{r}, u') = 1 | i^* = i] = \Pr[\mathsf{H}'_{i-1}(\mathcal{D}) = 1],$$

and consequently

$$\Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{u}, \mathbf{r}, u') = 1] = \sum_{i=1}^{q} \frac{1}{q} \Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{u}, \mathbf{r}, u') = 1 | i^* = i]$$

$$= \sum_{i=1}^{q} \frac{1}{q} \Pr[\mathsf{H}'_{i-1}(\mathcal{D}) = 1].$$

Putting all together, we get

$$\mathsf{Adv}_{\mathsf{DDLPN}}(\mathcal{D}_3) = |\Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{e}^\top \mathbf{A}, \mathbf{r}, \mathbf{e}^\top \mathbf{r}) = 1] - \Pr[\mathcal{D}_3(\mathbf{A}, \mathbf{u}, \mathbf{r}, u') = 1]|$$

$$= \left| \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathsf{H}'_i(\mathcal{D}) = 1] - \sum_{i=1}^{q} \frac{1}{q} \cdot \Pr[\mathsf{H}'_{i-1}(\mathcal{D}) = 1] \right|$$

$$= \frac{1}{q} |\Pr[\mathsf{H}'_k(\mathcal{D}) = 1] - \Pr[\mathsf{H}'_0(\mathcal{D}) = 1]|$$

$$= \frac{1}{q} |\Pr[\mathsf{Exp}_3(\mathcal{D}) = 1] - \Pr[\mathsf{Exp}_4(\mathcal{D}) = 1]|$$

$$\geq \epsilon_3/q.$$

Thus, $\mathcal{D}_3$ distinguishes $\mathsf{DDLPN}(n, 2n, 1, \rho)$ with advantage $\epsilon_3/q$.

We will now turn to the quantitative statement of the theorem. By the triangle inequality it holds that

$$\mathsf{Adv}_{\mathsf{DLPN}}(\mathcal{D}) \leq \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$\leq q \cdot \mathsf{Adv}_{\mathsf{eDLPN}}(\mathcal{D}_1) + \mathsf{Adv}_{\mathsf{DLPN}}(\mathcal{D}_2) + q \cdot \mathsf{Adv}_{\mathsf{DDLPN}}(\mathcal{D}_3).$$

This concludes the proof.

## 5 Conclusion

In this work we have constructed the first public key encryption scheme with KDM-CPA security for affine functions from the low-noise LPN assumption. Moreover, we have provided a novel connection between LPN with a bounded number of samples and LPN with an unbounded number of samples. Both results have analogues in the LWE realm (the KDM-CPA secure scheme of Applebaum et al. [10] and the LWE random self-reduction of Gentry et al. [26]). Both our results follow the same blueprint as their LWE counterparts. However, while in the LWE realm powerful statistical tools such as gaussian regularity [26] and the leftover-hash lemma [29,19] are available, no comparable statistical techniques are available in the LPN realm. Instead, our approach, following Kiltz et al.

[31] was to substitute these techniques with computational counterparts based on LPN. Specifically the extended LPN problem turned out to be very useful in filling this gap. A natural future direction in this line of work would be to try to lift further results from the LWE/SIS realm into the LPN realm, such as identity based encryption [17,3,4,7] or efficient and compact signature schemes [14,23,24].

## 6   Acknowledgement

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